A Quantum Unique Games Conjecture

Taro Spirig

QMATH, University of Copenhagen

Joint work with

Hamoon Mousavi

UC Berkeley

Main Motivation

How hard is it to compute or approximate the **quantum value** of nonlocal games? Undecidable in general [JNVWY 20] but little known apart from that.

Same question for **classical value** has been studied intensively. Can we use that knowledge?

$$\text{quantum value} \quad \overset{?}{\leftrightarrow} \quad \text{classical value}$$

Motivation: synchronous value

Quantum value of nonlocal games is hard to work with. Simpler yet as expressive to work with **synchronous quantum value**.

$$\text{quantum value} \quad \leftrightarrow \quad \text{synchronous quantum value} \quad \overset{?}{\leftrightarrow} \quad \text{classical value}$$

Motivation: Constraint Satisfaction Problems

Additional conceptual simplification: **CSPs** (defined in a few slides).

$$\text{Noncommutative CSP} \quad \overset{?}{\leftrightarrow} \quad \text{classical CSP}$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

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Question: Can we lift classical CSP techniques to noncommutative CSPs? Not easy (e.g. next talk).

(see however gadget reductions [Ji 13,Harris 23,Culf Mastel 24]).

This talk

Consider instead **quantum CSPs** (defined in a few slides).

$$\text{Noncommutative CSP} \quad \leftrightarrow \quad \text{quantum CSP} \quad \overset{?}{\leftrightarrow} \quad \text{classical CSP}$$

Here we can **”quantize” classical** results.

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Here we can **"quantize" classical** results.

Quantum CSPs correspond to **oracularizable synchronous quantum** strategies.

In summary

	Noncommutative CSP	$\leftrightarrow$	**quantum CSP**	this talk $\leftrightarrow$	**classical CSP**	
	$\Updownarrow$		$\Updownarrow$		$\Updownarrow$	

quantum value $\leftrightarrow$ synch quantum value $\leftrightarrow$ synch orac quantum value $\leftrightarrow$ classical value

Technical results

Indeed:

- **Quantum analog of PCP** theorem follows from [JNVWY 20],
- We propose a **quantum unique games conjecture (qUGC)**,
- We show **tight-hardness of approximation** for quantum analogs of 2-Lin and MaxCut from qUGC.

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Indeed:

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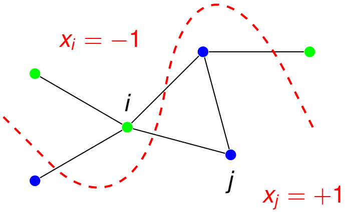
Remarks:

- Noncommutative UGC is not possible [KRT 10].
- Tight-hardness results are the same as classically only NP $\mapsto$ RE.
- Here ”quantum” CSPs do *not* refer to local Hamiltonian problems.

Quantum CSPs

Classical CSP: MaxCut

Instance of MaxCut is a simple graph:



Goal: assign color to vertices to maximize fraction of cut edges.

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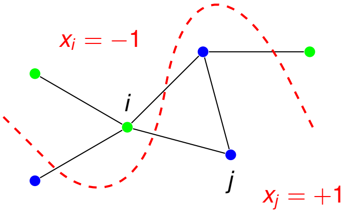
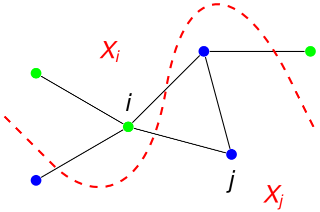


Goal: assign color to vertices to maximize fraction of cut edges.

$$\omega_c(G) = \max_x \frac{1}{|E|} \sum_{(i,j)\in E} \frac{1 - x_i x_j}{2}.$$

Noncommutative CSP: MaxCut

Operator generalization of classical coloring: **X_i finite dimensional Hermitian unitary**.

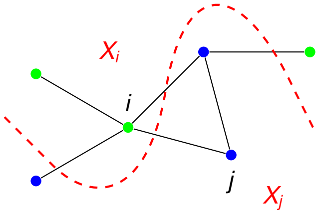


Goal: assign operators to vertices to maximize probability of cut edges.

$$\omega_{\mathrm{nc}}(G) = \sup_X \quad \frac{1}{|E|} \sum_{(i,j)\in E} \frac{1 - \operatorname{tr}\left(X_i X_j\right)}{2}$$

Quantum CSP: MaxCut

Same as Noncommutative MaxCut



With **additional requirement** that for $(i, j) \in E$:

$$[X_i, X_j] = 0.$$

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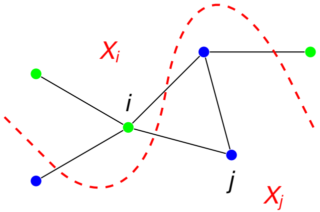


With **additional requirement** that for $(i, j) \in E$:

$$[X_i, X_j] = 0.$$

Terminology: quantum CSP because this requirement comes from **simultaneous measurability** (see also locally commuting algebras [HMPS17]).

Classical CSP Landscape

NP

Many problems are NP-hard, e.g. 3SAT, 3XOR or MaxCut. However, also have efficient ways to approximate them, e.g. Goemans-Williamson algorithm gives 0.878-approximation to MaxCut.

Approximation ratio: For any $G \in$ MaxCut, $\mathcal{A}_{GW}(G)$ outputs an assignment x such that

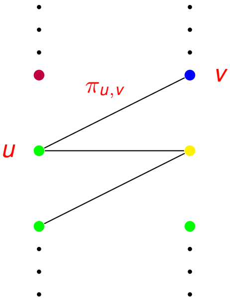
$$\omega_c(G; x) \geq \omega_c(G) \times 0.878.$$

Question: What is the best efficient approximation possible?

PCP

Starting point: Hardness of approximating Label-Cover

Label-Cover (LC) on alphabet $[m]$:



Goal: maximize fraction of satisfied constraints (modeled by $\pi_{u,v}$'s).

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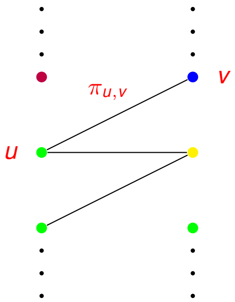

Goal: maximize fraction of satisfied constraints (modeled by $\pi_{u,v}$'s).

Note: as for MaxCut, can similarly define noncommutative and quantum LC.

PCP

Starting point: Hardness of approximating Label-Cover

PCP theorem

For every $\delta > 0$, there is large enough alphabet s.t. approximating Label-Cover within δ is NP-hard.

Reduction

Idea: Reduce PCP theorem to hardness of approximating other problems.

Reduction from A to B:

Efficient map $r : \mathrm{A} \ni G \mapsto G' \in \mathrm{B}$ such that if NP-hard to approximate A within δ-ratio then NP-hard to approximate B within γ-ratio.

Want: A=LC, B=MaxCut, any $\delta > 0$ and $\gamma = 0.878$.

Example: 3XOR

This idea works well for 3XOR, i.e. PCP theorem reduces to:

3XOR hardness of approximation [Håstad 01]

It is NP-hard to approximate 3XOR with ratio $> \frac{1}{2}$.

Tight result because random assignment is a $\frac{1}{2}$-approximation algorithm.

UGC

But it doesn’t work for MaxCut... We need to reduce from a tweaked version of PCP theorem.

Unique Games Conjecture [Khot 02]

For every $\delta > 0$, there is large enough alphabet s.t. approximating Unique-LC within δ is NP-hard.

UGC

But it doesn't work for MaxCut... We need to reduce from a tweaked version of PCP theorem.

Unique Games Conjecture [Khot 02]

For every $\delta > 0$, there is large enough alphabet s.t. approximating Unique-LC within δ is NP-hard.

Reduces to:

MaxCut hardness of approximation [KKMO 07]

Assuming UGC, it is NP-hard to approximate MaxCut with ratio > 0.878.

Quantum CSP Landscape

Quantum analog of PCP

By MIP* $=$RE, we *also* have:

Quantum analog of PCP

For every $\delta > 0$, there is large enough alphabet s.t. approximating LC_q within δ is RE-hard.

Quantum analog of PCP

By $\text{MIP}^* = \text{RE}$, we *also* have:

Quantum analog of PCP

For every $\delta > 0$, there is large enough alphabet s.t. approximating LC_q within δ is RE-hard.

As in classical case, Quantum PCP reduces to:

quantum 3XOR hardness [O'Donnel, Yuen 24]

It is RE-hard to approximate 3XOR_q with ratio $> \frac{1}{2}$.

Quantum UGC: Main Results

qUGC [this work]

For every $\delta > 0$, there is large enough alphabet s.t. approximating Unique-LC$_q$ within δ is RE-hard.

Quantum UGC: Main Results

qUGC [this work]

For every $\delta > 0$, there is large enough alphabet s.t. approximating Unique-LC$_q$ within δ is RE-hard.

And "quantize" other classical results quite easily.

2-LIN hardness of approximation [this work]

On a 2-LIN$_q$ instance with optimum $1 - \varepsilon$, it is RE-hard to find a $1 - O(\varepsilon^{\frac{1}{2}})$ approximation, assuming qUGC.

MaxCut hardness of approximation [this work]

Assuming wqUGC, it is RE-hard to approximate MaxCut$_q$ with ratio > 0.878.

Both are tight by a result of Goemans and Williamson.

Proof Ideas

Reduction

Reductions are complicated to come up with. Need to rely on **error-correcting codes** and smart ways to **test codes**, e.g. BLR test or Noise Stability.

Analysis is based on **Boolean Fourier analysis**.

Quantization

Reduction similar to classical. Analysis needs to be quantized.

Key ingredients:

- **Fourier:** Boolean Fourier analysis lifts to operator Fourier analysis. Classical probabilistic assignments become POVMs.
- **Commutativity**: simultaneously diagonalize some of the operators. Then can rely on classical analysis for eigenfunctions of these operators, e.g. we *do not need* operator generalization of Majority is Stablest (MIS) theorem.

Conclusion

Future Directions

$$\text{Noncommutative CSP} \overset{?}{\leftrightarrow} \text{quantum CSP} \overset{\text{this talk}}{\leftrightarrow} \text{classical CSP}$$

- **General phenomenon?** Can we easily quantize all classical reductions to the quantum setting?
- Weaker version of MIP* $=$RE? Could we directly **quantize Dinur's PCP** theorem?
- Can we lift similar ideas to the **noncommuatitve setting**?
- Which games are **oracularizable**? Do oracularizable strategies behave like synchronous strategies?

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Thank You!