Gap-preserving reductions and RE-completeness of independent set games

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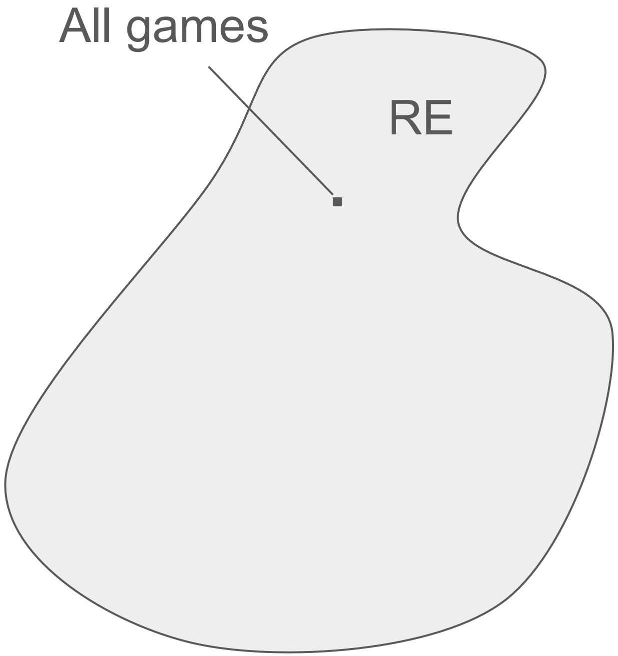

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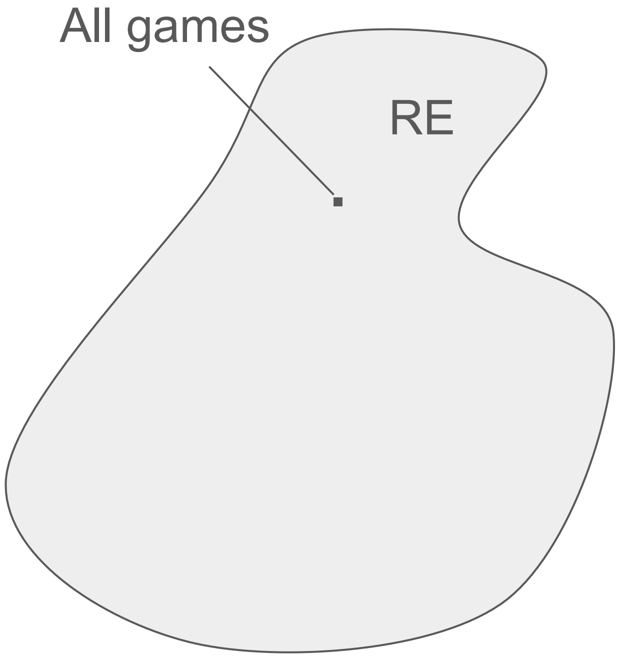
Classically: MIP=NEXP



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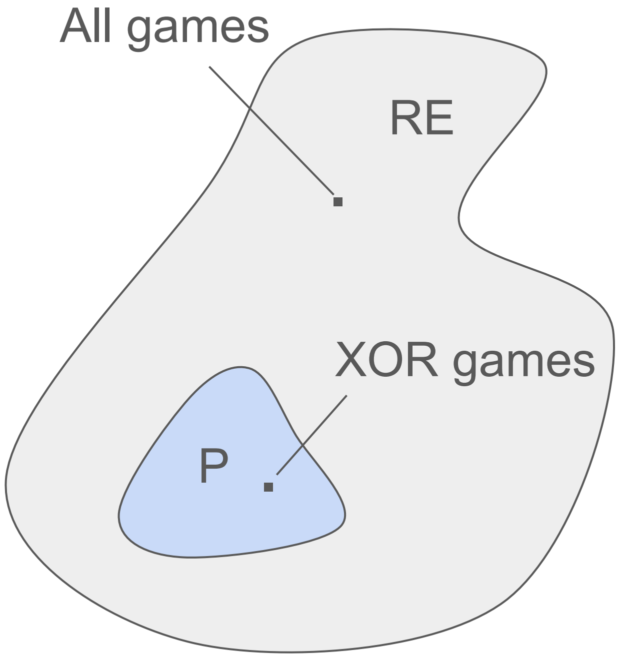
Answer: even approximating the quantum value of nonlocal games is in general **undecidable** (MIP*=RE [JNV+20]).

On the other hand, there are specific families of games for which computing or approximating the quantum value can be done **efficiently** [Tsi87, KRT10, Bei10, CMS24].



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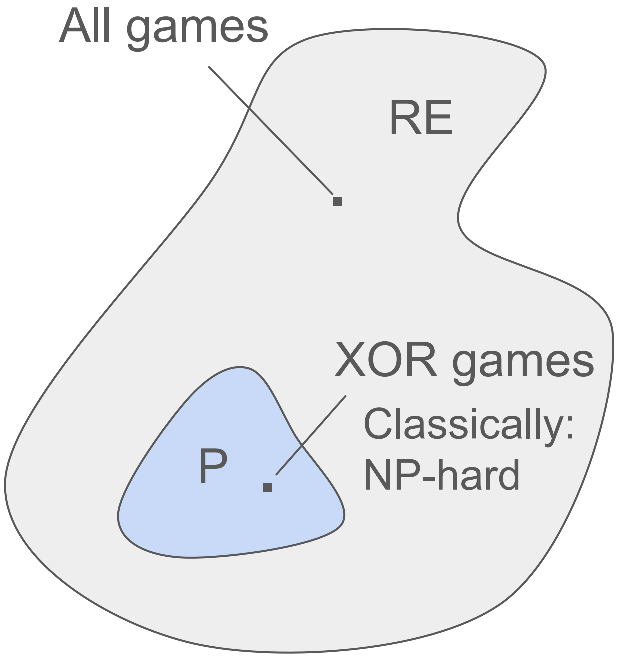
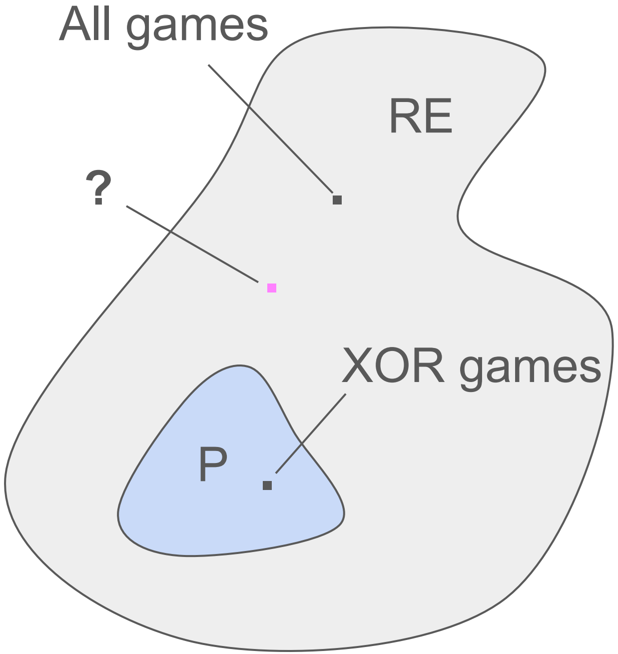


More refined questions

- What is the complexity of computing the quantum value of **natural well-structured families** of nonlocal games?
- Which of these families are MIP*/RE-complete (in a precise sense)?



More refined questions

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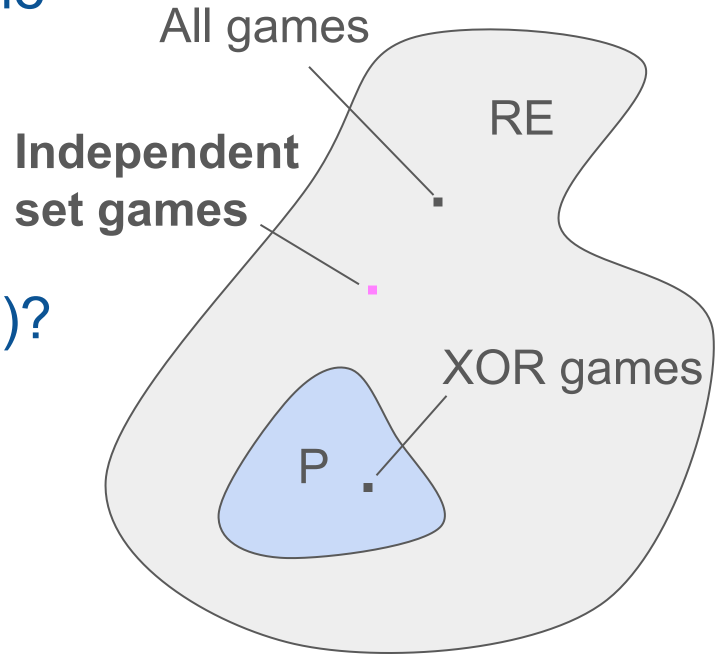
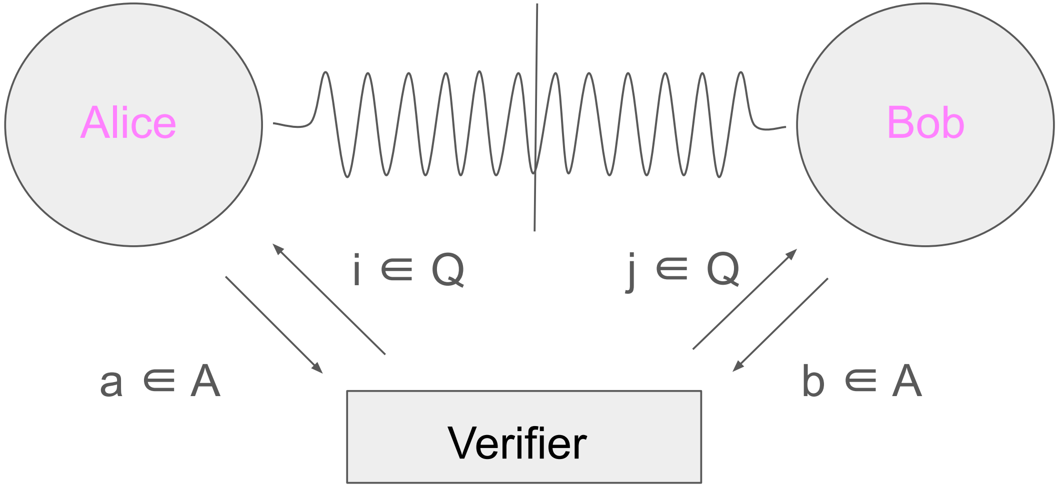
This work: Independent set games are MIP*/RE-complete.



Nonlocal Games - Definitions



Q questions set
A answers set

Goal for players: maximize probability of winning

Classical value: highest classical success probability

Quantum value: highest entangled success probability

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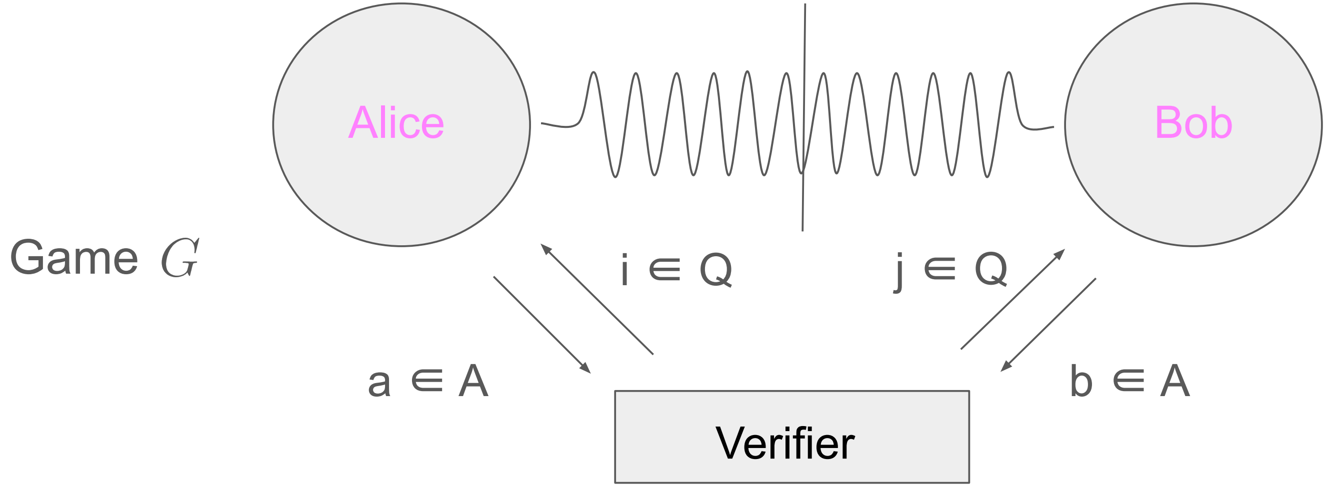

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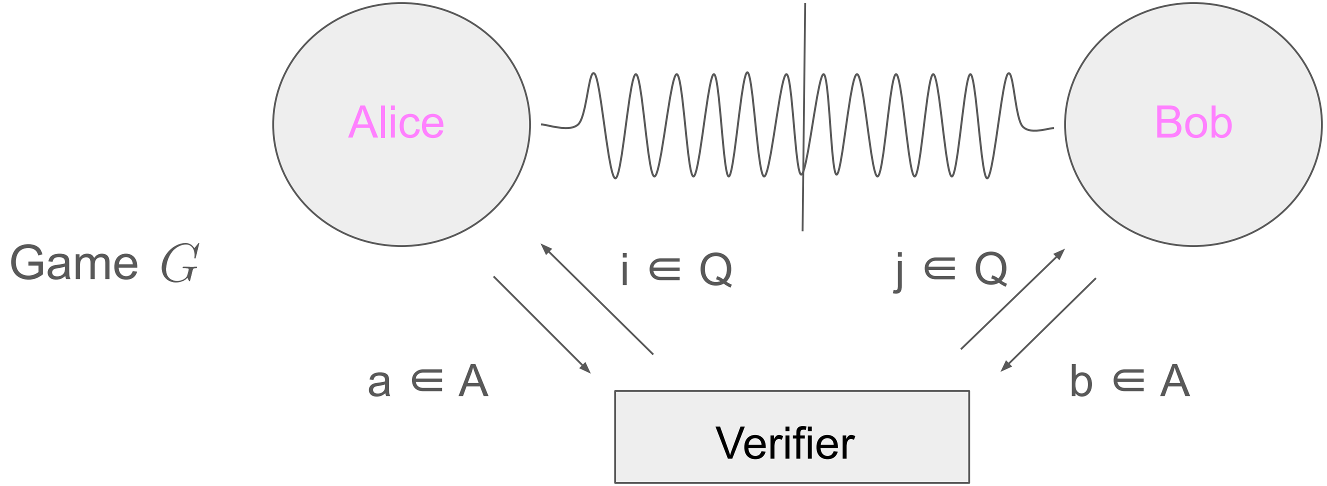
Goal for players: maximize probability of winning

$\omega(G)$ Classical value: highest classical success probability

$\omega^*(G)$ Quantum value: highest entangled success probability

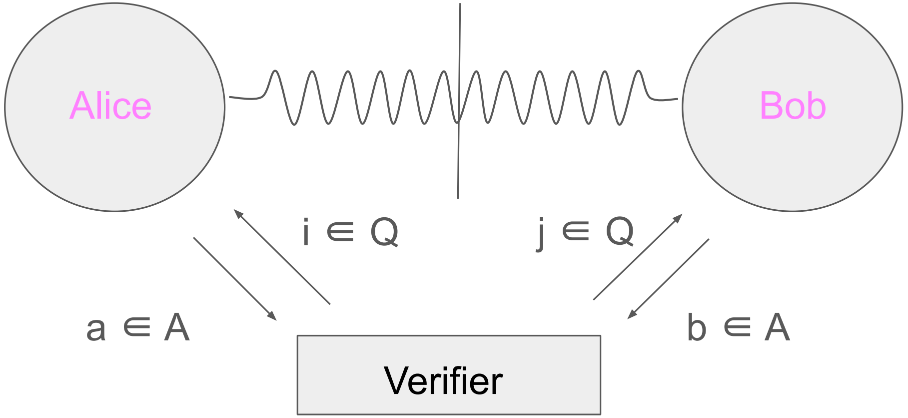
Nonlocal Games - Motivation

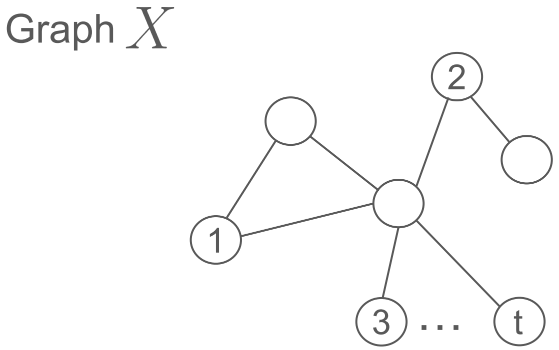


There are games where $\omega(G) < \omega^*(G)$

Applications e.g. delegation of quantum computation.

Nonlocal Games - Independent Set Games





t-Independent Set Game for graph X [MRV15]:

$\mathrm{Q} = [t] = \{1, \ldots, t\}$ $\mathrm{A} = \mathcal{V}(X)$

Rules: (consistency check) if i = j then players must respond a=b
(independence test) if i $\neq$ j, then players must respond with distinct non-adjacent vertices

Complexity - Gapped promise problems

Consider family of nonlocal games $C = \{\mathcal{G}_n\}_n$.

Definition: A (c, s)-gap* promise problem for this family is:

Given a nonlocal game $\mathcal{G} \in C$, decide whether:

- $\omega^*(\mathcal{G}) \geq c$ or
- $\omega^*(\mathcal{G}) < s$.

Similar to set up for hardness of approximation.

Complexity - MIP*=RE

MIP*=RE [JNV+20]:

For any $0 < s < 1$, there exists a family of games for which the $(1, s)$-gap* problem is RE-complete.

The family of games considered here is the family of synchronous games.
Our goal is to find a more structured family.

Our result

Theorem [MSSV25]

There exist a constant $0 < s < 1$, a positive integer t, and a family of t-independent set games C such that the $(1, s)$-gap* problem for C is RE-complete.

Note: classically the same problem is easy for fixed t.

Technique - Gap-preserving reduction

Definition: A reduction from a (c, s)-gap* problem for family $C = \{G\}$ to the (c', s')-gap* problem for family $C' = \{G'\}$ is an efficient mapping $G \mapsto G'$ satisfying

- (completeness) if $\omega^*(G) \geq c$, then $\omega^*(G') \geq c'$, and
- (soundness) if $\omega^*(G) < s$, then $\omega^*(G') < s'$.

Then if (c, s)-gap* for C is RE-hard, (c', s')-gap* for C' is also RE-hard.

Again similar set up as in hardness of approximation.

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Caveat: throughout this talk I am omitting that family of games are succinctly presented. This makes things a bit more complicated.

Our gap-preserving reduction

Lemma [MSSV25]

Let G be a synchronous game with t questions.

We construct a graph X_G whose t-independent set game G' satisfies:

- (completeness) if $\omega^*(G) = 1$, then $\omega^*(G') = 1$,
- (soundness) if $\omega^*(G) < 1 - \varepsilon$, then $\omega^*(G') < 1 - \kappa \frac{\varepsilon^8}{t^4}$,

where κ is a universal constant.

Our gap-preserving reduction

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To get our main theorem, need to rely on refinement of MIP*=RE by [NZ23] who construct an RE-complete family of games with **constant** number of questions.

Technical tool - new stability theorem

Need to round operators which almost satisfy certain relations to operators which satisfy these relations exactly.

Stability theorem [MSSV25]

Let $0 < \varepsilon < 1$. Consider projections $\{P_1, \ldots, P_m\}$ that almost sum to the identity:

$$\left\| \sum_{j=1}^{m} P_j - 1 \right\|_2 \leq \varepsilon$$

Then there exist projections $\{Q_1, \ldots, Q_m\}$ such that

$$\sum_{j=1}^{m} Q_j = 1 \quad \text{and} \quad \sum_{j=1}^{m} \|P_j - Q_j\|_2^2 \leq \mathcal{O}(\varepsilon)$$

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Stability theorem [MSSV25]

Almost PVM

Let $0 < \varepsilon < 1$. Consider projections $\{P_1, \ldots, P_m\}$ that almost sum to the identity:

$$\left\| \sum_{j=1}^{m} P_j - 1 \right\|_2 \leq \varepsilon$$

Actual PVM

Then there exist projections $\{Q_1, \ldots, Q_m\}$ such that

No m-dependence!

$$\sum_{j=1}^{m} Q_j = 1 \quad \text{and} \quad \sum_{j=1}^{m} \|P_j - Q_j\|_2^2 \leq \mathcal{O}(\varepsilon)$$

Summary

- We make progress in understanding the complexity of computing the quantum value of natural well-structured families of nonlocal games.
- In particular: Independent Set games are RE-complete even for fixed t.
- Method: (c, s)-gap* problems and gap-preserving reductions.
- Technical tool: a new stability theorem.

Comparison to [CM24] (previous talk)

- [CM24] shows RE-completeness of several classically NP-hard problems.
- [CM24] uses different techniques and reduces from RE-complete problem with constant *answer set*.

Summary

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- In particular: Independent Set games are RE-complete even for fixed t.
- Method: (c, s)-gap* problems and gap-preserving reductions.
- Technical tool: a new stability theorem.

Open questions

- For t fixed, how does the complexity of $(1, s)$-gap* problem for t-Independent set games vary with s?
- Understanding the complexity landscape of nonlocal games is burgeoning: this work, MS24, CM24, MS25, TV25, CDVZ25, CMPS25.
- Need more “good” stability theorems?

References

[JNV+20] Zhengfeng Ji, Anand Natarajan, Thomas Vidick, John Wright, Henry Yuen. *MIP*=RE*.

[Tsi87] Boris Tsirelson. *Quantum analogues of the bell inequalities. The case of two spatially separated domains.*

[KRT10] Julia Kempe, Oded Regev, Ben Toner. *Unique games with entangled provers are easy.*

[Bei10] Salman Beigi. *A lower bound on the value of entangled binary games.*

[CMS24] Eric Culf, Hamoon Mousavi, Taro Spirig. *Approximation Algorithms for Noncommutative CSPs.*

[MRV15] Laura Mančinska, David E. Roberson, Antonios Varvitsiotis. *On deciding the existence of perfect entangled strategies for nonlocal games.*

[CM24] Eric Culf, Kieran Mastel. *RE-completeness of entangled constraint satisfaction problems.*

[MS24] Kieran Mastel, William Slofstra. *Two Prover Perfect Zero Knowledge for MIP*.*

[MS25] Hamoon Mousavi, Taro Spirig. *A quantum unique games conjecture.*

[TV25] Aviv Taller, Thomas Vidick. *Approximating the quantum value of an LCS game is RE-hard.*

[CDVZ25] Eric Culf, Josse van Dobben de Bruyn, Matthijs Vernooij, Peter Zeman. *Existence and nonexistence of commutativity gadgets for entangled CSPs.*

[CMPS25] Eric Culf, Kieran Mastel, Connor Paddock, Taro Spirig. *The quantum smooth label cover problem is undecidable.*