Geometric Quantisation of the n-dimensional Harmonic Oscillator

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Semester Project in Theoretical Physics

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Spring 2019

Acknowledgment

I would like to deeply thank my supervisor Dr. Gabriele Rembado for his incredible support during this project and for sharing his passion for this beautiful subject with me.
I would also like to thank Prof. Giovanni Felder for giving me the opportunity to do this project and for the meetings we had.

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Chapter 1

Introduction

Geometric quantisation is the process of constructing a quantum mechanical system corresponding to a classical system via methods of differential geometry. This geometric approach of quantisation provides more insight than deformation quantisation as it enables one to construct the phase space of the quantum theory as well.

A classical phase space is the set of pure states of a system. Mathematically the classical phase space is defined by a symplectic manifold, which is a smooth manifold with an additional piece of structure, a so called symplectic form. Classical observables, such as position or energy, are smooth functions defined on the classical phase space. The symplectic form fixes the dynamics of the classical observables, via Poisson brackets.

A quantum phase space is built out of a Hilbert space, the so called quantum Hilbert space, equipped with a representation of the Weyl algebra. A pure state corresponds to a unitary vector of the quantum Hilbert space (up to a phase). These states are called wave functions and they encode the physical states of the system. The quantum analogue to classical observables are (self-adjoint) operators acting on this Hilbert space, the quantum operators. The commutator of the quantum operators, the analogue of the Poisson brackets in the classical system, encodes the dynamics of the quantum mechanical system, in the Heisenberg picture.

There is a family of quantum theories parameterised by the variable $\hbar$, the quantum of action. The correspondence principle tells us that in the classical limit, i.e. when $\hbar$ goes to zero, the classical system corresponding to the quantum one should be recovered. Quantisation gives a way to find the quantum theories corresponding to a given classical theory. In other words, the goal of quantisation is to construct a quantum Hilbert space with quantum operators corresponding to classical observables, which follow the same dynamics, i.e. the Poisson brackets should agree with the commutator in the classical limit. One possibility to achieve this goal is given by geometric quantisation, which we shall describe in the following.

We first need to make sure that geometric quantisation is applicable. Indeed, not all classical systems are quantisable. We therefore need some conditions to hold on the given classical phase space, the so called prequantisation criteria. Intuitively these conditions already impose a notion of discreteness, crucial in quantum mechanics. If the classical phase space fulfills the prequantisation criteria, we can proceed with prequantisation. This means constructing a Hilbert space out of sections of a prequantum line bundle. The Hilbert space, thus constructed, is however too big, because its elements can depend on both position and momentum coordinates of the symplectic manifold, which is forbidden by the Heisenberg uncertainty principle from quantum mechanics. We therefore need to restrict this Hilbert space. Geometric quantisation imposes the use of polarisation to do this. A polarisation naturally leads to local choices of canonical coordinates and we can thus choose for example sections which are only dependent on the position coordinates. We get the desired quantum Hilbert space out of the completion of the space of polarised sections, where one may try to construct quantum operators.

As an illustration, we will work out in full details the geometric quantisation of a harmonic oscillator of arbitrary dimensions. For this, we will use two different types of polarisation, real and Kähler polarisations. This results in two different representations of the Weyl algebra, which are unitary equivalent according to the Stone–von Neumann theorem. We prove this explicilty by constructing the Segal–Bargmann transform and showing that it intertwines the two representations.

A particular effort has been spent in order to make this work self-contained. Chapter 2 is therefore dedicated to a thorough differential geometry background.

Chapter 2

Differential Geometry Background

2.1 Disclaimer

This section should be seen as a review of the most important objects which are used in geometric quantisation. We will sometimes define objects which are more general than the ones used in the examples of this work. These objects are however crucial to understand more complicated examples. Therefore, one could understand the rest of this work by neglecting the sections concerning principal bundles, frame and associated bundles, principal bundle connection as well as connections on frame and associated bundles. Most of the propositions in this section are going to be left without proof (sometimes a sketch of the proof will be given) since these are often quite involved and would make the content of this work too long. All the proofs can be found in [1] or in [2].

2.2 Smooth Manifolds and their Tangent Spaces

In this chapter, we will discuss a few of the most essential objects of differential geometry. We will first define smooth manifolds and their tangent spaces and finally we will discuss vector fields which are smooth maps from the smooth manifold to the tangent space.

2.2.1 Smooth Manifolds

For the following definitions X is a topological space.

Definition 2.2.1. We call X Hausdorff if for every pair $x \neq y$ of points in X there are open subsets $U, V \subseteq X$ such that $x \in U$, $y \in V$ and $U \cap V = \varnothing$.

Definition 2.2.2. A collection of open subsets of X is denoted $\{U_a \mid a \in A\}$ for A some index set. Such an open subset is called an open cover if $X = \cup_{a \in A} U_a$.

Definition 2.2.3. An open cover $\{U_a \mid a \in A\}$ is said to be locally finite if for every $x \in X$ there exists a neighbourhood W of x such that the set $\{a \in A \mid U_a \cap W \neq \emptyset\}$ is finite. A refinement is another open cover $\{V_b \mid b \in B\}$ such that for every $b \in B$ there exists $a \in A$ with $V_b \subseteq U_a$.

Definition 2.2.4. The topological space X is said to be paracompact if every open cover has a locally finite refinement.

Definition 2.2.5. A homeomorphism is a continuous map between two topological spaces whose inverse is also continuous.

Definition 2.2.6. The topological space X is said to be locally Euclidean of dimension n if for every point $x \in X$ there exists a neighbourhood U of x, an open set $O \subseteq \mathbb{R}^n$, and a homeomorphism $\sigma : U \to O$.

Definition 2.2.7. A topological manifold of dimension n is a topological space M with the following properties:

1. it is locally Euclidean of dimension n
2. it is Hausdorff and has at most countably many connected components
3. it is paracompact.

Definition 2.2.8. Let $O_a \subseteq \mathbb{R}^n$ and $O_b \subseteq \mathbb{R}^m$ open sets with $n, m \in \mathbb{N}$. A differentiable map $f : O_a \to O_b$ is of class C^r if it is r-times differentiable and the differential of order r is continuous. We say that f is smooth or of class C^∞ if f is of class C^r for every $r \geq 1$. If f is both smooth and bijective and the inverse function is also smooth then we say that f is a diffeomorphism.

Definition 2.2.9. Let M be a topological manifold of dimension n and $\{U_a \mid a \in A\}$ an open cover of M. For each $a \in A$, we define O_a as an open set in $\mathbb{R}^n$ and $\sigma_a : U_a \to O_a$ a homeomorphism such that the following compatibility condition is satisfied. Suppose $a, b \in A$ are such that $U_a \cap U_b \neq \emptyset$. Then the composition

$$\sigma_b \circ \sigma_a^{-1}|_{\sigma_a(U_a \cap U_b)} : \sigma_a(U_a \cap U_b) \to \sigma_b(U_a \cap U_b)$$

should be a diffeomorphism. A smooth atlas on M is then the collection:

$$\Sigma = \{\sigma_a : U_a \to O_a \mid a \in A\}$$

The map $\sigma_a : U_a \to O_a$ is called a chart of the atlas Σ and the composition $\sigma_b \circ \sigma_a^{-1}|_{\sigma_a(U_a \cap U_b)}$ is called a transition map. We will denote an atlas Σ by the tuple (U_a, σ_a).

Remark 2.2.10. We will from now on drop the restriction for the transition maps for convenience, i.e. we will only write $\sigma_b \circ \sigma_a^{-1}$.

Definition 2.2.11. A smooth structure on a topological manifold is an equivalence class of smooth atlases, where two smooth atlases Σ_1 and Σ_2 are equivalent if their union is also a smooth atlas.

Remark 2.2.12. A smooth structure contains a unique maximal atlas being the union of all the atlases in the equivalence class. We will then normally work with the preferred representative of the class, namely the maximal atlas.

Finally we can define what a smooth manifold is, in the following way.

Definition 2.2.13. A smooth manifold of dimension n is a pair (M, Σ) where M is a topological manifold of dimension n and Σ is a smooth structure on M.

Example 2.2.14. We cite a few important examples of smooth manifolds:

- Any vector space V of dimension k is a smooth manifold. One can show that a vector space is a topological manifold and the atlas contains one single chart which maps the whole vector space to $\mathbb{R}^k$. Indeed one can choose a basis $B = (v_1, ..., v_k) \subseteq V$. Then the chart can be defined by setting $\sigma : V \to \mathbb{R}^k,\ v_i \mapsto e_i$ for any $i \in \{1, ..., k\}$ and $(e_i)_{i \in 1,...,k}$ the standard basis of $\mathbb{R}^k$. There is obviously no compatibility condition to check.

- Not every subset of a vector space is a smooth manifold. Take $\mathbb{R}^2$ with $V \subseteq \mathbb{R}^2$ the subset composed of two lines crossing each other at the origin. Assume that we have a chart σ mapping a neighbourhood of the crossing to $\mathbb{R}$. If one takes the origin away, V gets split into four different connected components. However removing $\sigma(0)$ from $\mathbb{R}$ will result in only two connected components. This leads to a contradiction since a chart should be a homeomorphism.

- The unit sphere $\mathbb{S}^n \coloneqq \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$. First, we recall the definition of compactness. A topological space is called compact if it is Hausdorff and each of its open cover has a finite subcover. The sphere $\mathbb{S}^n$ is a compact smooth manifold of dimension n. We can indeed define a smooth atlas in the following way. We denote the poles of the sphere with $x_N \coloneqq (0, ..., 0, 1)$ and $x_S \coloneqq (0, ..., 0, -1)$. If $n = 2$, x_N is the north pole and x_S is the south pole of the 2-sphere. Let U_N be the sphere $\mathbb{S}^n$ without x_N and U_S be the sphere $\mathbb{S}^n$ without x_S, i.e. $U_N \coloneqq \mathbb{S}^n \setminus \{x_N\}$ and $U_S \coloneqq \mathbb{S}^n \setminus \{x_S\}$ two open subsets of $\mathbb{S}^n$. Then $\{U_N, U_S\}$ be an open cover of $\mathbb{S}^n$. We can then define the charts $\sigma_N : U_N \to \mathbb{R}^n,\ (x_1, ..., x_{n+1}) \mapsto \frac{1}{1-x_{n+1}}(x_1, ..., x_n)$ and $\sigma_S : U_S \to \mathbb{R}^n,\ (x_1, ..., x_{n+1}) \mapsto \frac{1}{1+x_{n+1}}(x_1, ..., x_n)$. These are the stereographic projections from the poles. The transition maps are given by $\sigma_N \circ \sigma_S^{-1} = \sigma_S \circ \sigma_N^{-1} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\},\ (y_1, ..., y_n) \mapsto \frac{1}{\sum_{i=1}^n (y_i)^2}(y_1, ..., y_n)$. Indeed, one can check that $(y_1, ..., y_n) \mapsto \frac{1}{\sum_{i=1}^n (y_i)^2}(y_1, ..., y_n)$ is a diffeomorphism, because the denominator of higher order derivatives is only vanishing at $0 \in \mathbb{R}^n$.

- Every open subset U of a manifold M is a manifold. One can indeed check that U can be equipped with the topology and the atlas induced by M.

For the following definitions we will denote M for a smooth manifold of dimension n and x a point in M. We will use the letter O for open subsets of $\mathbb{R}^n$ and U for open subsets of smooth manifolds. Finally, we will denote with $\sigma : U \to O$ a chart on M.

Definition 2.2.15. Let $f : O \subseteq \mathbb{R}^n \to \mathbb{R}^k$ be a smooth map as defined in Definition 2.2.8. Then $f_i \coloneqq u^i \circ f$ is the ith component of f, where u^i is the canonical projection on the ith factor. We denote the Jacobian matrix of f at a point x as $Df(x)$, the $k \times n$ matrix whose (i, j)th entry is $\frac{\partial f_i}{\partial x_j}(x) \coloneqq \frac{\partial}{\partial x_j}(u^i \circ f)$ the partial derivative of f_i, where x_j denotes the canonical coordinates. A column of the matrix $Df(x)$ is denoted by $D_j f(x)$ and the (i, j)th entry of $Df(x)$ is denoted by $D_j(u^i \circ f)(x)$.

Definition 2.2.16. Let $\phi : M \to N$ a continuous map between two smooth manifolds. Let $x \in M$ be a point with U_a a neighbourhood of x and U_b a neighbourhood of $\phi(x)$. Let $\sigma : U_a \to O_a$ be any chart on M and $\tau : U_b \to O_b$ be any chart on N. Then ϕ is called a smooth map if the composition:

$$\tau \circ \phi \circ \sigma^{-1} : \sigma(U_a \cap \phi^{-1}(U_b)) \to \tau(\phi(U_a) \cap U_b)$$

is of class C^∞ as in Definition 2.2.8. If ϕ is smooth and bijective and the inverse function $N \to M$ is also smooth then ϕ is said to be a diffeomorphism.

Remark 2.2.17. We denote the space of all the smooth maps from M to N with $C^\infty(M, N)$. If $N = \mathbb{R}$ we write $C^\infty(M)$ instead of $C^\infty(M, \mathbb{R})$.

For a chart $\sigma : U \to O$ on M, one can show that $u^i \circ \sigma$ is a smooth function from U to $\mathbb{R}$.

Definition 2.2.18. For $x \in M$ and σ defined on a neighbourhood of x, we denote the function $u^i \circ \sigma \in C^\infty(U)$ with x_i. We say that x_i are the local coordinates about x in the chart σ.

2.2.2 Tangent Spaces

Now we will move on to tangent spaces, which will help us define the tangent map $D\phi$ of a smooth map $\phi : M \to N$, for M, N smooth manifolds. The tangent map is the best linear approximation of ϕ around x. We will give two viewpoints of tangent spaces: one abstract definition via derivations of the $\mathbb{R}$-algebra $C^\infty(M)$ and another more concrete one via curves on the smooth manifold.

Definition 2.2.19. A derivation of $C^\infty(M)$ at x is a linear map $w : C^\infty(M) \to \mathbb{R}$ which satisfies the derivation property (Leibniz) $w(fg) = f(x)w(g) + g(x)w(f)$ for $f, g \in C^\infty(M)$.

Definition 2.2.20. The tangent space of M at $x \in M$ is the space of derivations at x as in Definition 2.2.19. We denote it by $T_x M$.

Proposition 2.2.21. The space of derivations at x is a vector space.

Proof. Let $v, w \in T_xM$ and $f, g \in C^\infty(M)$. We only verify that $v + w$ is also a derivation, since the other conditions follow directly:

$$\begin{aligned}(v+w)(fg) &= v(fg) + w(fg)\\ &= f(x)v(g) + g(x)v(f) + f(x)w(g) + g(x)w(f)\\ &= f(x)(v+w)(g) + g(x)(v+w)(f).\end{aligned}$$

□

We will now denote with v a tangent vector of T_xM.

Remark 2.2.22. The derivation defined above depends on the chosen local neighbourhood around x. A more elegant way of defining the tangent space would be a definition which doesn't depend on the choice of the neighbourhood. This is possible with germs of smooth functions which are equivalence classes of smooth functions on a neighbourhood of x. Two smooth functions defined on neighbourhoods of x are equivalent if they coincide on an open subset of the intersection of their domain. With such a definition one still has the information about the intrinsic local behaviour around x of the smooth functions but without specifying the neighbourhood around x. The interested reader can find more about this topic in [1].

Definition 2.2.23. Let x be any point in $U \subseteq M$ an open set. Define the map $\frac{\partial}{\partial x_i}\big|_x$ as:

$$\frac{\partial}{\partial x_i}\bigg|_x : C^\infty(U) \to \mathbb{R}\ ,\ \frac{\partial}{\partial x_i}\bigg|_x(f) := D_i(f \circ \sigma^{-1})(\sigma(x))$$

Proposition 2.2.24. The map $\frac{\partial}{\partial x_i}\big|_x$ of Definition 2.2.23 is a derivation.

Proof. It is clear that it is $\mathbb{R}$-linear. Let $f, g \in C^\infty(M)$. Then we have:

$$\begin{aligned}\frac{\partial}{\partial x_i}\bigg|_x(fg) &= D_i(fg \circ \sigma^{-1})(\sigma(x))\\ &= \frac{(\partial fg) \circ \sigma^{-1}}{\partial x_i}(\sigma(x)) + \frac{(f\partial g) \circ \sigma^{-1}}{\partial x_i}(\sigma(x))\\ &= g(x)\frac{\partial}{\partial x_i}\bigg|_x(f) + f(x)\frac{\partial}{\partial x_i}\bigg|_x(g)\end{aligned}$$

where we used Definition 2.2.15 to write $D_i(fg \circ \sigma^{-1})(\sigma(x))$ in the usual form of partial derivative. □

Proposition 2.2.25. Let $x \in U$. The ordered set $(\frac{\partial}{\partial x_1}|_x, ..., \frac{\partial}{\partial x_n}|_x)$ forms a basis of T_xM. Hence the dimension of T_xM equals the dimension of M.

Definition 2.2.26. Let M and N be smooth manifolds and let $\phi : M \to N$ be a smooth map. Fix $x \in M$ and $v \in T_xM$. Let $w \in T_{\phi(x)}N$ be the tangent vector which satisfies $w(f) := v(f \circ \phi)$, $\forall f \in C^\infty(N)$. We denote $D\phi(x) : T_xM \to T_{\phi(x)}N$ thus defined $v \mapsto w = D\phi(x)[v]$. We call $D\phi(x)$ the tangent map of ϕ at x.

Proposition 2.2.27. The tangent map $D\phi(x)$ is a linear map.

Proof. Let $\lambda_1, \lambda_2 \in \mathbb{C}$ and $v_1, v_2 \in T_xM$.

$$\begin{aligned} D\phi(x)[\lambda_1 v_1 + \lambda_2 v_2] &= (\lambda_1 v_1 + \lambda_2 v_2)(f \circ \phi) \\ &= \lambda_1 (v_1)(f \circ \phi) + \lambda_2 (v_2)(f \circ \phi) \\ &= \lambda_1 D\phi(x)[v_1] + \lambda_2 D\phi(x)[v_2] \end{aligned}$$

□

Remark 2.2.28. The other possible definition of the tangent space uses the notion of a smooth curve in a smooth manifold.
A smooth curve in a smooth manifold M is a smooth map $\gamma : (a, b) \to M$ where the interval $(a, b) \subseteq \mathbb{R}$ is an open subset of $\mathbb{R}$ equipped with the canonical smooth structure. Let $\varepsilon \in \mathbb{R}_{>0}$. A velocity vector at $x \in M$ is an equivalence class of smooth curves $\gamma : (-\varepsilon, \varepsilon) \to M$ such that $\gamma(0) = x$. The equivalence relation is the following: γ is equivalent to δ if $(\sigma \circ \gamma)'(0) = (\sigma \circ \delta)'(0)$, where σ is some chart centered about x and $'$ is the usual derivative of real functions. We denote the space of all velocity vectors at $x \in M$ with $T_x^v M$ which one can show to be a vector space.
The map $T_x^v M \mapsto T_xM,\ [\gamma] \mapsto \gamma'(0)$ is well-defined and is an isomorphism. In words, we can associate to each smooth curve $\gamma : (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = x$ a tangent vector in T_xM by taking its velocity vector. The velocity vector is indeed a derivation on $C^\infty(M)$ by setting $\gamma'(t)(f) \coloneqq (f \circ \gamma)'(t),\ f \in C^\infty(M)$. One can check that it is well-defined by verifying that for two curves in the same equivalence class $\gamma, \delta \in [\gamma]$, $(f \circ \gamma)'(0) = (f \circ \delta)'(0)$ for any $f \in C^\infty(M)$.
We now understand the intuition behind tangent vectors. For example the velocity vector of a curve on the unit sphere $\mathbb{S}^2$ gives a vector tangent to the sphere (in the usual sense).

Proposition 2.2.29. Let M be a smooth manifold with the smooth atlas $\Sigma = (U_a, \sigma_a)$ with the chart $\sigma_a \colon U_a \to O_a \subseteq \mathbb{R}^n$. Consider the disjoint union of the tangent spaces:

$$TM \coloneqq \bigsqcup_{x \in M} T_xM.$$

The space TM can be endowed with the structure of a smooth manifold induced by the smooth structure of M.

Proof. We first show that we can define a topology on TM. We declare $TM|_{U_a} \coloneqq \bigsqcup_{x \in U_a} T_xM$ to be open. This defines a toplogy on TM. Indeed $TM|_{U_a} \cap TM|_{U_b} = \bigsqcup_{x \in U_a \cap U_b} T_xM = TM|_{U_a \cap U_b}$ is again open. Similarly, $\bigcup_{i \in I} TM|_{U_i} = TM|_{\bigcup_{i \in I} U_i}$ for some index set I is open. Now we show that there is an atlas induced by the atlas Σ of M. Let x_a^i be the local coordinates associated to the local chart σ_a. For $p \in TM|_{U_a} = \bigsqcup_{x \in U_a} T_xM$, there is a unique $x \in U_a$ such that $p = (x, v)$ with $v \in T_xM$. Since $v \in T_xM$ we can write it as a linear combination $v = \sum_i \lambda_i \frac{\partial}{\partial x_a^i}|_x,\ \lambda_i \in \mathbb{R}$. We define the map $\widetilde{\sigma}_a \colon TM|_{U_a} \to O_a \times \mathbb{R}^n$ by setting $\widetilde{\sigma}_a(p) = \widetilde{\sigma}_a(x, \sum_i \lambda_i \frac{\partial}{\partial x_a^i}|_x) \coloneqq (\sigma_a(x), (\lambda_1, ..., \lambda_n))$. The map $\widetilde{\sigma}_a$ is a local chart because σ_a is a local chart for M and a change of basis in $\mathbb{R}^n$ is given by multiplication with invertible matrices. This shows that $\widetilde{\Sigma} = (TM|_{U_a}, \widetilde{\sigma}_a)$ is a smooth atlas. $\square$

Definition 2.2.30. The tangent bundle is the disjoint union of tangent spaces TM endowed with the structure of a smooth manifold from Proposition 2.2.29. We denote an element of TM as a pair (x, v) to indicate that $v \in T_xM$. There is a smooth map $\pi : TM \to M$ given by $\pi(x, v) = x$.

We will actually understand later that the tangent bundle TM is an example of a vector bundle (and one of the most important examples). Vector bundles are going to be central in the process of geometric quantisation.

Before discussing vector fields we now give a few important definitions which require the tangent map. Immersed submanifold are going to be useful when defining distributions and embedded manifolds are going to be used when discussing fibre bundles.

Definition 2.2.31. A smooth map $f : M \to N$ between two smooth manifolds is called a submersion if its derivative is surjective.

Definition 2.2.32. Let $\phi : M \to N$ be a smooth map between smooth manifolds. We say that ϕ is an immersion if the linear map $D\phi(x)$ is injective for every $x \in M$. If in addition ϕ itself is injective then we say that ϕ is an injective immersion. If in addition ϕ maps M homeomorphically on to $\phi(M)$ we say that ϕ is an embedding.

Definition 2.2.33. Let M and N be smooth manifolds. We say that M is an immersed submanifold of N if the inclusion $\imath : M \hookrightarrow N$ is an immersion. We denote the immersed submanifold with $(M, \imath)$.

Definition 2.2.34. Let M and N be smooth manifolds with $M \subseteq N$. We say that M is an embedded submanifold of N if the inclusion $M \hookrightarrow N$ is an embedding.

2.2.3 Vector Fields

We now arrive to the definition of a vector field which are smooth maps taking values in the tangent bundle.

Definition 2.2.35. Let $U \subseteq M$ be an open set and π the map from Definition 2.2.30. A vector field X on U is a smooth map $X : U \to TM|_U$, s.t. $\pi \circ X = \mathrm{id}$. We denote by $\Omega^0(U, TM)$ the set of all vector fields on U. In other words a vector field defines a smoothly varying family of vectors over the open set U.

Remark 2.2.36. Setting $\frac{\partial}{\partial x_i}(x) \coloneqq \frac{\partial}{\partial x_i}\big|_x$ defines a vector field on U denoted $\frac{\partial}{\partial x_i}$. We can then write any vector field X in local coordinates as $X(x) = \sum_i X_i(x) \frac{\partial}{\partial x_i}\big|_x$ with $X_i \in C^\infty(U)$.

Remark 2.2.37. The index 0 in the notation $\Omega^0(U,TM)$ will make sense later on when we discuss differential forms. We will also understand that vector fields are actually sections of the tangent bundle.

Definition 2.2.38. Let $f \in C^\infty(U)$. Then for any $x \in U$ and any vector field X on U we can define a function $X.f : U \to \mathbb{R}$ with $X.f(x) := X(x)f$, which is well defined since $X(x) \in T_xM$ is a derivation on $C^\infty(M)$. In local coordinates we can write $X.f(x) = \sum_i X_i(x)\frac{\partial f}{\partial x_i}(x)$, with the same $X_i \in C^\infty(U)$ as in Remark 2.2.49.

Proposition 2.2.39. The map $X.f$ is smooth.

The space of sections of the tangent bundle, i.e. vector fields, carry an additional structure, the Lie bracket.

Definition 2.2.40. A Lie algebra is a vector space $\mathfrak{g}$ endowed with an alternating bilinear operation called the Lie bracket

$$\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \ (v,w) \mapsto [v,w],$$

which satisfies the Jacobi identity

$$[u,[v,w]] + [v,[w,u]] + [w,[u,v]] = 0, \ \forall u,v,w \in \mathfrak{g}.$$

Proposition 2.2.41. The space of vector fields $\Omega^0(U,TM)$ equipped with the Lie bracket defined by $[X,Y] = X \circ Y - Y \circ X, \ \forall X,Y \in \Omega^0(U,TM)$ is a real Lie algebra. This Lie bracket is called the commutator.

Remark 2.2.42. With Definition 2.2.38 we can write $([X,Y]).f$ for any $X,Y \in \Omega^0(U,TM)$ and $f \in C^\infty(U)$. Explicitly we can write $([X,Y]).f = X.(Y.f) - Y.(X.f)$ and one can show that this is still a derivation.

2.2.4 Cotangent Spaces

To the tangent space we can associate its dual vector space: the cotangent space. Cotangent spaces can be assembled to cotangent bundle in the same way as tangent bundle.

Definition 2.2.43. The cotangent space of M at x is the dual vector space $(T_xM)^* = \mathrm{Hom}_\mathbb{R}(T_xM,\mathbb{R})$. We will denote it by T_x^*M.

Definition 2.2.44. Let U be a neighbourhood of x and let $f \in C^\infty(U)$. Then f defines an element $df|_x \in T_x^*M$, the differential of f at x, by $df|_x(v) := v(f), \ v \in T_xM$.

Proposition 2.2.45. Suppose σ is a chart defined on U and x_i the local coordinates about x in the chart σ. The set $(dx_i|_x)_{i=1,\dots,n}$ is a basis of T_x^*M.

Proof. The set $(dx_i|_x)_{i=1,\dots,n}$ is the dual basis to $(\frac{\partial}{\partial x_i}|_x)_{i=1,\dots,n}$ since

$$dx_j|_x\left(\frac{\partial}{\partial x_i}\Big|_x\right) = \frac{\partial}{\partial x_i}\Big|_x(x_j) = \delta_i^j.$$

□

Just as before for TM one can show that the disjoint union of cotangent spaces can be equipped with a smooth structure.

Proposition 2.2.46. Let M be a smooth manifold with the smooth atlas $\Sigma = (U_a, \sigma_a)$ with the chart $\sigma_a \colon U_a \to O_a \subseteq \mathbb{R}^n$. Consider the disjoint union of the tangent spaces:

$$T^*M \coloneqq \bigsqcup_{x \in M} T_x^*M.$$

The space T^*M can be endowed with the structure of a smooth manifold induced by the smooth structure of M.

Definition 2.2.47. The cotangent bundle is the disjoint union of tangent spaces T^*M endowed with the structure of a smooth manifold from Proposition 2.2.46. We denote an element of T^*M as a pair (x, λ) to indicate that $\lambda \in T_x^*M$. There is a smooth map $\pi : T^*M \to M$ given by $\pi(x, \lambda) = x$.

Analogous to the Definition of vector fields we can define differential 1-forms. We will define later on differential r-forms in greater generality.

Definition 2.2.48. Let $U \subseteq M$ be an open set and π the map from Definition 2.2.47. A differential 1-form α on U is a smooth map $\alpha : U \to T^*M|_U$, s.t. $\pi \circ \alpha = \mathrm{id}$. We denote by $\Omega^0(U, T^*M)$ the set of differential 1-forms on U. In other words a differential 1-form defines a smoothly varying family of dual vectors over the openset U.

Remark 2.2.49. Setting $dx_i(x) \coloneqq dx_i|_x$ defines a differential 1-form on U denoted dx_i. We can then write any differential 1-form α in local coordinates as $\alpha(x) = \sum_i \alpha_i(x) dx_i|_x$ with $\alpha_i \in C^\infty(U)$. In particular, for a smooth function $f \in C^\infty(U)$ setting $df(x) \coloneqq df|_x$ defines a differential 1-form on U and in local coordinates we can write it as $df(x) = \sum_i \frac{\partial f}{\partial x_i}(x) dx_i|_x$.

2.2.5 Distributions

Definition 2.2.50. Let M be a smooth manifold of dimension n and let $k \leq n$ an integer. A smooth distribution Δ on M of dimension k is a choice of k-dimensional linear subspaces $\Delta_x \subseteq T_xM$ for each $x \in M$ that vary smoothly with x in the following sense. For each point $x_0 \in M$ there exists a neighbourhood U of x_0 and k vector fields $X_1, .., X_k \in \Omega^0(U, TM)$ such that

$$\Delta_x = \mathrm{span}_{\mathbb{R}}\{X_1(x), ..., X_k(x)\}, \ \forall x \in U.$$

Distributions are going to be used to define connections on principal bundles. The following definitions are going to be particularly relevant when defining polarisations.

Definition 2.2.51. Let Δ be a k-dimensional distribution on M, and suppose $(L, \imath)$ is a k-dimensional immersed submanifold. We say that L is an integral manifold of Δ if

$$D\imath(x)[T_xL] = \Delta_x, \ \forall x \in L.$$

Definition 2.2.52. A distribution Δ is said to be integrable if $\forall x \in M$ there exists $(L, \imath)$ such that L is an integral manifold of Δ.

Definition 2.2.53. Let Δ be a distribution on M and let X be a vector field on M. We say that X belongs to Δ (or that it is tangent to Δ) if $X(x) \in \Delta_x$ for each $x \in M$.

Definition 2.2.54. We denote with $[\cdot,\cdot]$ the Lie bracket from Definition 2.2.41. A distribution Δ is said to be involutive if $[X, Y]$ belongs to Δ whenever X and Y belong to Δ. Thus an involutive distribution is one for which the space of vector fields belonging to it forms a Lie subablgebra of $\Omega^0(M, TM)$.

Proposition 2.2.55. Integrable Distributions are involutive.

The converse is false in general but it is true in the constant rank case as the following theorem shows.

Theorem 2.2.56. *(Frobenius Theorem) Let Δ be an integrable distribution on M. If Δ has constant rank (equivalently one could say that Δ is a vector subbundle of the tangent bundle, c.f. section 2.4) and is involutive then Δ is integrable.*

Definition 2.2.57. A k-dimensional foliation $\mathcal{F}$ of M is a partition into k-dimensional connected immersed submanifolds, called leaves of the foliation, such that:

1. The collection of tangent spaces to the leaves defines a distribution Δ on M.
2. Any connected integral manifold of Δ is contained in some leaf of $\mathcal{F}$.

Each leaf L of $\mathcal{F}$ is a maximal connected integral manifold of Δ.

We now understand that foliations $\mathcal{F}$ are equivalent to integrable distributions. Indeed, by definition a foliation defines an integrable distribution and given a distribution one can construct a foliation by defining its leaves as being the maximal connected integral manifolds of the distribution Δ.

2.3 Lie Groups

In this section, we define Lie groups, which are smooth manifolds with group structure. We also discuss representations and we define especially the adjoint representation of a Lie group.

Definition 2.3.1. A Lie group G is a smooth manifold endowed with a group structure in the following sense. It fulfils the properties that the group multiplication $G \times G \to G,\ m(a,b) = ab$ and group inversion $G \to G,\ i(a) = a^{-1}$ are both smooth maps, where $G \times G$ is given the natural smooth structure of a direct product.

Proposition 2.3.2. Let G be a Lie group. The tangent space to G at the identity element T_eG is a Lie algebra in the sense of Definition 2.2.40. We denote this Lie algebra with $\mathfrak{g} := T_eG$.

Example 2.3.3. We give here a few of the most important examples of Lie groups for our purpose.

1. The set of invertible $n \times n$ matrices $\mathrm{GL}_n(\mathbb{R})$ is a Lie group under matrix multiplication. The Lie algebra associated to it is $\mathfrak{gl}(n)$, the space of all square matrices $\mathrm{Mat}(n)$ with the Lie bracket given by the commutator.

2. The subset of orthogonal Matrices $\mathrm{O}(n) \subseteq \mathrm{GL}_n(\mathbb{R})$ is defined as $\mathrm{O}(n) = \{A \in \mathrm{GL}_n(\mathbb{R}) \mid AA^T = I_n\}$, where I_n is the identity element. The subset $\mathrm{O}(n)$ is closed in $\mathrm{GL}_n(\mathbb{R})$ and is (therefore) a Lie subgroup (see closed subgroup theorem, Theorem 9.11 in [1]). The Lie algebra $\mathfrak{o}(n)$ of $\mathrm{O}(n)$ can be identified by $\{A \in \mathfrak{gl}(n) \mid A + A^T = 0\}$, where A^T denotes the transpose of A. This result comes from the implicit function theorem (see [1]). Let $\mathrm{Sym}(n)$ be the subset of symmetric matrices. If we define a map $\phi : \mathrm{Mat}(n) \to \mathrm{Sym}(n),\ A \mapsto A^TA - I_n$, its derivative $D\phi$ is surjective at the identity matrix I_n: $D\phi(I_n)[A] = A + A^T$. Using the implicit function theorem, it follows that $T_{I_n}O(n) = \ker D\phi(e)$. In other words $\mathfrak{o}(n)$ is the Lie Algebra of antisymmetric matrices with the commutator as Lie bracket.

3. The subset of special unitary matrices $\mathrm{SU}(2) \subseteq \mathrm{GL}_2(\mathbb{C})$ is defined as $\mathrm{SU}(2) = \{A \in \mathrm{GL}_2(\mathbb{C}) \mid AA^\dagger = I_n,\ \det(A) = 1\}$, where $A^\dagger := \bar{A}^T$ the conjugate transpose of A. It is a Lie subgroup of $\mathrm{GL}_2(\mathbb{C})$ because it is also closed in $\mathrm{GL}_2(\mathbb{C})$. One can describe the Lie algebra of $\mathrm{SU}(2)$ with the same method as above. The Lie algebra $\mathfrak{su}(2)$ can be identified by $\{A \in \mathrm{GL}_2(\mathbb{C}) \mid A^\dagger + A = 0,\ \mathrm{tr}(A) = 0\}$. In other words $\mathfrak{su}(2)$ is the Lie algebra of skew-Hermitian traceless matrices with the commutator as Lie bracket.

We can now discuss representations of Lie groups and Lie algebras.

Definition 2.3.4. A representation of a group G on a finite vector space V is a continuous homomorphism $\rho\colon G \to \mathrm{GL}(V)$. We denote a representation by the tuple (ρ, V).

This definition holds in particular if G is a Lie group.

Definition 2.3.5. A representation of a Lie algebra $\mathfrak{g}$ on a finite vector space V is a continuous homomorphism $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$.

The adjoint representation is the representation of a Lie group G on its Lie algebra.

Definition 2.3.6. The map $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$, $\big(\mathrm{Ad}(g)\big)(x) = gxg^{-1}$ is a representation of the Lie group G. We call it the adjoint representation of G.

A particularly important notion for a representation is irreducibility that we now define.

Definition 2.3.7. An invariant subspace of a representation (ρ, V) is a subspace $W \subseteq V$ given such that $\rho(g)W \subseteq W$ for all $g \in G$. A representation (ρ, V) is irreducible if there are no invariant subspace other than V and $\{0\}$.

2.4 Bundles

In this chapter we define fibre bundles and we then specialize to vector/line bundles and principal bundles. We then introduce associated bundles, which are vector bundles associated to principal bundles. We then discuss sections of vector bundles and we finally discuss Hermitian vector bundles.

2.4.1 Fibre Bundles

Definition 2.4.1. Let E, M and F be smooth manifolds and suppose $\pi : E \to M$ is a smooth surjective map. We say that the tuple (π, E, M, F) is a fibre bundle over M with fibre F if for every point $x \in M$ there exists a neighbourhood U of x and a smooth map $\alpha : \pi^{-1}(U) \to F$ such that:

$$\varphi := (\pi, \alpha) : \pi^{-1}(U) \to U \times F \tag{2.1}$$

is a diffeomorphism. We call φ a local trivialisation, (U, φ) a system of local trivialisations and α a bundle chart. We call E the total space of the bundle, M the base space and F the fibre. The collection of bundle charts is called a bundle atlas.

Definition 2.4.2. Given a fibre bundle E over M with fibre F we set $E_x := \pi^{-1}(x)$ for $x \in M$ and call E_x the fibre over x.

Lemma 2.4.3. Let $\pi : E \to M$ be a fibre bundle with fibre F. Then π is a submersion, and moreover each fibre E_x is an embedded submanifold of E which is diffeomorphic to F.

Proof (Sketch). Let (U, φ) be a system of local trivialisations. The tangent map $D\pi$ is a composition of submersions, namely φ and the first projection pr_1. Using the implicit function theorem, E_x is an embedded submanifold. Finally the composition of the second projection pr_2 after (π, α) is a diffeomorphism from E_x to F. □

Definition 2.4.4. Let $\alpha : \pi^{-1}(U_\alpha) \to F$ and $\beta : \pi^{-1}(U_\beta) \to F$ be two bundle charts. We define the transition function:

$$\rho_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \mathrm{Diff}(F),\ \ \rho_{\alpha\beta}(x) := \alpha|_{E_x} \circ \beta|_{E_x}^{-1}$$

Thus if $p \in \pi^{-1}(U_\alpha \cap U_\beta)$ one has:

$$\alpha(p) = \Big[\rho_{\alpha\beta}(\pi(p))\Big](\beta(p)).$$

If $\gamma \colon \pi^{-1}(U_\gamma) \to F$ is another bundle chart then the cocycle condition is satisfied, namely:

$$\rho_{\alpha\gamma}(x) = \rho_{\alpha\beta} \circ \rho_{\beta\gamma}(x),\ \ \forall x \in U_\alpha \cap U_\beta \cap U_\gamma$$

Definition 2.4.5. Let F be a smooth manifold and suppose we are given a left action $\mu : G \times F \to F$ of a Lie group G on F. We say that μ is effective if $\mu_a = \mathrm{Id}_F$ implies $a = e_G$.

Definition 2.4.6. Suppose $\pi : E \to M$ is a fibre bundle with fibre F, and suppose G is a Lie group acting effectively from the left on F. We say that two bundle charts $\alpha : \pi^{-1}(U) \to F$ and $\beta : \pi^{-1}(U) \to F$ are (G, μ)-compatible if there exists a smooth map $\widetilde{\rho}_{\alpha\beta} : U \cap V \to G$ such that

$$\Big(\rho_{\alpha\beta}(x)\Big)(z) = \mu(\widetilde{\rho}_{\alpha\beta}(x), z),\ \forall x \in U \cap V, \forall z \in F$$

We say that the fibre bundle E has structure group G if all the charts of the bundle atlas are (G, μ)-compatible. We say that the bundle atlas is a (G, μ)-bundle atlas.

Remark 2.4.7. In particular, if $F = \mathbb{R}^k$ is a vector space and $G \subseteq \mathrm{GL}_k(\mathbb{R})$ is a Lie group then the action will always be taken to be the standard one. Thus we write that the bundle charts are G-compatible omitting the action μ.

Example 2.4.8. We now give an example of fibre bundle. Let M denote the smooth manifold $\mathbb{C}P^1 = \mathbb{C}^2\backslash\{0\}/\mathbb{C}^\times$, E denote the smooth manifold $\mathbb{C}^2\backslash\{0\}$ and F the smooth manifold $\mathbb{C}^\times$. One can check that $\pi : E \to M$ canonically defined is a fibre bundle with fibre F.

Definition 2.4.9. Let $\pi_i \colon E_i \to M_i,\ i \in \{1, 2\}$ denote two fibre bundles. Suppose we are given two smooth maps $\Phi \colon E_1 \to E_2$ and $\phi \colon M_1 \to M_2$. We say that Φ is a vector bundle morphism along ϕ if the restriction to each $E_1|_x$ is a linear map from $E_1|_x$ to $E_2|_{\phi(x)}$. In other words Φ is fibre-preserving. If Φ maps each fibre $E_1|_x$ isomorphically onto $E_2|_{\phi(x)}$ then Φ is called a vector bundle isomorphism along ϕ. If ϕ is the identity map $\mathrm{id} \colon M \to M$ we call Φ a fibre bundle homorphism.

2.4.2 Vector Bundles

We now move on to vector bundles which are special types of fibre bundles. Vector bundles are going to be central in the geometric quantisation process we are going to follow, since we are going to construct the quantum Hilbert phase space as a space of sections of vector bundles.

Definition 2.4.10. A vector bundle of rank k is a fibre bundle $\pi : E \to M$ whose fibre is $F = \mathbb{R}^k$ and whose structure group $G \subseteq \mathrm{GL}_k(\mathbb{R})$ is a Lie subgroup. A line bundle is a vector bundle of rank 1.

For clarity we rewrite the definition of a fibre bundle in the case of a vector bundle.

Definition 2.4.11. Let $\pi : E \to M$ be a surjective map between two smooth manifolds and set $E_x \coloneqq \pi^{-1}(x)$. We say that E is a vector bundle of rank k if each E_x has the structure of a k-dimensional vector space, and any $x \in M$ has a neighbourhood U together with a smooth map $\alpha : \pi^{-1}(U) \to \mathbb{R}^k$ such that:

1. $\varphi \coloneqq (\pi, \alpha) : \pi^{-1}(U) \to U \times \mathbb{R}^k$ is a diffeomorphism

2. $\alpha|_{E_x} : E_x \to \mathbb{R}^k$ is an isomorphism of vector spaces.

The change of charts is given by $\mathrm{GL}_k(\mathbb{R})$ matrices. The bundle charts are $\mathrm{GL}_k(\mathbb{R})$-compatible.

Example 2.4.12. The trivial rank k vector bundle over M is defined by $E := M \times F = M \times \mathbb{R}^k$ with $\pi : M \times \mathbb{R}^k \to M$ the first projection pr_1. Then U can be M and we define $\alpha : M \times \mathbb{R}^k \to \mathbb{R}^k$ to be the second projection pr_2. The transition map is then the Identity matrix.

Definition 2.4.13. A vector bundle is said to be trivialisable if it is isomorphic to the rank k trivial bundle.

Example 2.4.14. Another relevant example is the tangent bundle from Definition 2.2.30. Indeed, we can define $\pi \colon TM \to M$ as a vector bundle of rank n, where n is the dimension of M. For $\sigma_\alpha \colon U_\alpha \to O_\alpha$ a chart on M with local coordinates x_i we can define bundle charts with $\alpha : \pi^{-1}(U_\alpha) \to \mathbb{R}^n,\ \alpha(x, v) = \sum_i dx_i|_x(v)e_i$, where e_i is the canonical basis of $\mathbb{R}^n$. For $\sigma_\beta : U_\beta \to O_\beta$, the transition map from Definition 2.4.4 becomes $\rho_{\alpha\beta} = D(\sigma_\alpha \circ \sigma_\beta^{-1})(\sigma_\beta(x)) \in \mathrm{GL}_n(\mathbb{R}) \subseteq \mathrm{Diff}(\mathbb{R}^n)$.

Example 2.4.15. Let $\phi : M \to N$ be a smooth map and suppose $\pi \colon E \to N$ is a fibre bundle. Then $M \times E \to M$ is a trivial bundle. The pullback bundle ϕ^*E is defined as follows. We define

$$\phi^*E := \{(x, p) \in M \times E \mid \phi(x) = \pi(p)\},$$

with projections $\mathrm{pr}_1 : \phi^*E \to M$ and $\mathrm{pr}_2 : \phi^*E \to E$. The fibre $\phi^*(E)$ over $x \in M$ is $\{x\} \times E_{\phi(x)}$ which is diffeomorphic to $E_{\phi(x)}$ under pr_2. If $\alpha \colon \pi^{-1}(U) \to F$ is a bundle chart for E then $\alpha \circ \mathrm{pr}_2 \colon \mathrm{pr}_1^{-1}(\phi^{-1}(U)) \to F$ is a bundle chart for ϕ^*E. Thus ϕ^*E is a fibre bundle. Moreover, one can show that if (E, M, F, G) is a vector bundle then the pullback bundle is a vector bundle with structure group a Lie subgroup of G.

We will now state the Metatheorem which gives a taste as to why vector bundles are so important.

Theorem 2.4.16. *(Metatheorem) Any construction we can perform with vector spaces without choosing a basis, we can also perform with vector bundles.*

Example 2.4.17. A few examples for the Metatheorem are the following: if $\pi_{E,F} \colon E, F \to M$ are vector bundles of dimension n and m respectively then we can form another vector bundle $E^* \to M$ called the dual bundle, whose fibre over x is defined as $(E^*)_x := (E_x)^*$. We can define a tensor vector bundle $E \otimes F$ and an exterior product vector bundle $E \wedge F$. Finally, We can also construct a direct sum vector bundle $E \oplus F$, where $(E \oplus F)|_x = (E|_x \oplus F|_x)$. Let us specify a bit the vector bundle structure of $E \oplus F$. The other example's structures are very similar. Choose the system of local trivialisations for E as being $\{(U_a, \phi_a^E)\}_{a \in I}$ and for F as being $\{(U_a, \phi_a^F)\}_{a \in I}$ for some index set I.

Then one can check that $\pi_{E\oplus F}\colon E\oplus F\to M,\ (p_E,p_F)\mapsto(\pi_E(p_E),\pi_F(p_F))$ is a vector bundle with the system of local trivialisations $\{(U_a,(\phi_a^E,\phi_a^F))\}_{a\in I}$ and transition maps $\rho_{\alpha,\beta}^{E\oplus F}: U_\alpha\cap U_\beta\to \mathrm{GL}_{m+n}(\mathbb{R})$ of the form

$$\rho_{\alpha,\beta}^{E\oplus F}=\begin{pmatrix}\rho_{\alpha,\beta}^{E} & 0\\ 0 & \rho_{\alpha,\beta}^{F}\end{pmatrix}.$$

2.4.3 Principal Bundles

We will now look at another very important type of fibre bundle: principal bundles. Principal bundles are related to vector bundles, because any principal bundle we can associate a vector bundle and vice versa. There is actually a one-to-one correspondence between vector bundles of rank k over M and principal $\mathrm{GL}_k(\mathbb{R})$-bundles.

Definition 2.4.18. A right action of a Lie group G on a smooth manifold M is a smooth map $\mu: M\times G\to M,\ (x,a)\mapsto x\cdot a$ satisfying $\mu(x,ab)=\mu(\mu(x,a),b),\ \mu(x,e_G)=x$ for all $a,b\in G$ and $x\in M$. The action is

- free if $\mu(x,a)=x$ for some $x\in M$ and $a\in G$ implies $a=e$. In words, an action is free if $a\in G,\ \mu(\cdot,a)\colon M\to M$ has a fixpoint if and only if $a=e_G$.
- transitive if for any $x,y\in M$ there exists an $a\in G$ so that $\mu(x,a)=y$.

For all principal bundles we will use a different notation for a right action μ, as defined above. We will instead define a right action as being the map $\cdot : (p,a)\mapsto p\cdot a$ for $a\in G$.

Definition 2.4.19. Let $\pi: P\to M$ be a fibre bundle with fibre a Lie group G. Assume that there exists a free fibre-preserving right action of G on P and a bundle atlas for P with the property that each bundle chart $\alpha:\pi^{-1}(U)\to G$ is G-equivariant. The G-equivariance property means that:

$$\alpha(p\cdot a)=\alpha(p)a,\ \forall p\in\pi^{-1}(U),\ \forall a\in G$$

Then we say that P is a principal bundle over M with group G or a G-principal bundle.

Let us fix a principal G-bundle P for the rest of this section.

Lemma 2.4.20. The structure group of P as a fibre bundle is G itself, where we let G act on itself via left translation.

Definition 2.4.21. Given the right action from Definition 2.4.18. The G-orbit of $p\in P$ is the set $\{p\cdot a\mid a\in G\}$. We denote the G-orbit of p with $p\cdot G$.

Lemma 2.4.22. The fibres of $\pi\colon P\to M$ are the orbits for the action of G on P and hence M is diffeomorphic to the quotient space of P/G. In particular, G acts transitively on the fibres.

Proof (Sketch). Let us first prove that the fibre over x is contained in a G-orbit for all x. This implies that the action is transitive on fibres. Let $x \in M$, $p, q \in P_x$ and $\alpha : \pi^{-1}(U) \to G$ be a bundle chart over a neighbourhood U of x. Denote $a \coloneqq \alpha(p)$ and $b \coloneqq \alpha(q)$. Then we have

$$(\pi, \alpha)(p \cdot a^{-1}b) \overset{(1)}{=} (x, \alpha(p)a^{-1}b) = (x, b) = (\pi, \alpha)(q)$$

where we used in (1) that G is fibre-preserving and α is G-equivariant. Hence $q = p \cdot a^{-1}b$, since (π, α) is a diffeomorphism. So we have shown that $P_x \subseteq p \cdot G,\ \forall p \in G$. Conversely, $p \cdot G \subseteq P_x$ for $\pi(p) = x$ follows from the fact that G is fibre preserving. Thus $P_x = p \cdot G$.
Now consider the quotient space $P/G = \{p \cdot G \mid p \in P\}$. One can show that P/G has the structure of a smooth manifold and the natural bijection $p \cdot G \to \pi(p)$ is a diffeomorphism. □

2.4.4 Frame Bundles and Associated Bundles

We now define frame bundles. This construction shows one direction of the one-to-one correspondence between rank k vector bundles and principal $\mathrm{GL}_k(\mathbb{R})$-bundle.

Definition 2.4.23. Let $\pi : E \to M$ be a vector bundle. A frame at x for E is an isomorphism $\mathbb{R}^k \to E_x$. We denote the set of frames at x with $\mathrm{Fr}(E_x)$.

Remark 2.4.24. One can show that the set of frame $\mathrm{Fr}(E_x)$ from Definition 2.4.23 is equivalent to the set of all ordered basis of E_x.

Definition 2.4.25. Consider the disjoint union of the frames:

$$\mathrm{Fr}(E) \coloneqq \bigsqcup_{x \in M} \mathrm{Fr}(E_x)$$

with $\widehat{\pi} : \mathrm{Fr}(E) \to M$ the map that sends $\mathrm{Fr}(E_x)$ to x. The frame bundle is the disjoint union of the frames $\mathrm{Fr}(E)$ equipped with the structure of a $\mathrm{GL}_k(\mathbb{R})$-bundle over M.

Frame bundles are the principal $\mathrm{GL}_k(\mathbb{R})$-bundles associated to vector bundles of rank k.

Remark 2.4.26. Intuitively, $\mathrm{Fr}(E)$ is a $\mathrm{GL}_k(\mathbb{R})$-bundle because there is exactly one element of $\mathrm{GL}_k(\mathbb{R})$ to change basis. Therefore each set of frame $\mathrm{Fr}(E_x)$ is in bijection with $\mathrm{GL}_k(\mathbb{R})$ up to choosing a point.

Associated bundles give the other direction of the one-to-one correspondence between vector bundles of rank k and principal $\mathrm{GL}_k(\mathbb{R})$-bundles. It associates vector bundles to principal bundles.

Definition 2.4.27. Let $\pi : P \to M$ be a principal G-bundle and assume G acts effectively on another manifold F on the left via $\rho : G \times F \to F$. Define an equivalence relation $\sim$ on $P \times F$ by:

$$(p \cdot a, v) \sim (p, \rho(a, v)),\ p \in P,\ a \in G,\ v \in F.$$

Define $P \times_G F$ to be the quotient space $(P \times F)/\sim$. Writing $[p, v]$ for the equivalence class of (p, v), we define a map $\widetilde{\pi} : P \times_G F \to M$ by setting $\widetilde{\pi}[p, v] = \pi(p)$.

Proposition 2.4.28. The tuple $(\widetilde{\pi}, P \times_G F, M, F)$ is a fibre bundle over M with fibre F and structure group G.

Proposition 2.4.29. Let $\pi : P \to M$ be a principal G-bundle and V be a vector space. Suppose $\rho : G \to \mathrm{GL}(V)$ is a smooth effective representation of G on V. Then $(\widetilde{\pi}, P \times_G V, M, V)$ is a vector bundle over M according to Definition 2.4.27. Additionally, for any $p \in P$, the map $L_p : V \to (P \times_G V)|_{\pi(p)},\ v \mapsto [p, v]$ is an isomorphism. Thus for $x \in M$ the vector space structure on $(P \times_G V)|_x$ is given by

$$[p, v] + r[p, w] = L_p(v + rw) = [p, v + rw].$$

We call $P \times_G V$ the associated vector bundle to P over M.

Remark 2.4.30. If we take ρ to be the tautological representation of $\mathrm{GL}_k(\mathbb{R})$ on $\mathbb{R}^k$ then the association $P \to P \times_{\mathrm{GL(k)}} \mathbb{R}^k$ defines an inverse to the frame bundle construction. Note that principal bundles are more general than vector bundles because there exist Lie groups which are not matrix group.

2.4.5 Sections of Bundles

Now that we understand fibre bundles, vector bundles and principal bundles, we can define smooth maps from the base space to the total space of a fibre bundle.

Definition 2.4.31. Let $\pi : E \to M$ be a fibre bundle. A global section of E is a smooth map $s : M \to E$ such that $\pi \circ s = \mathrm{Id}_M$, i.e. $s(x) \in E_x,\ \forall x \in M$. The set of global sections is denoted $\Omega^0(M, E)$. A local section of E on an open set $U \subseteq M$ is a section of the bundle $E|_U \to U$ of E. We denote by $\Omega^0(U, E)$ the set of all local sections over U.

As mentioned earlier the quantum Hilbert phase space that we are going to construct will be a space of sections on vector bundles.

Remark 2.4.32. The notation $\Omega^0(M, E)$ should ring a bell. Recall that the set of all vector fields on $U \subseteq M$ was noted $\Omega^0(U, TM)$. This is sensible because vector fields are by definition sections of the tangent bundle (cf. Example 2.4.14).

Example 2.4.33. In the case of the trivial bundle $L = M \times \mathbb{R}$ the space $\Omega^0(M, L)$ is naturally isomorphic to the space $C^\infty(M)$. We can indeed define a map $C^\infty(M) \to \Omega^0(M, L),\ f \mapsto fs_1$ where $s_1 \in \Omega^0(M, L)$ is defined via $s_1(x) \coloneqq (x, 1),\ x \in M$. It is naturally an isomorphism since for any $x \in M$ we have $fs_1(x) = f(x)s_1(x) = f(x)(x, 1) = (x, f(x))$ and any section is of this form. We call the smooth function $f \in C^\infty(M)$ a coefficient function.

Proposition 2.4.34. The vector bundle $\pi : L \to M$ is trivialisable if and only if it admits a global nowhere vanishing section of L.

Example 2.4.35. In case of a non-necessarily trivial line bundle L we have an open cover $\{U_j\}_{j\in J}$ of M with local trivialisations $\varphi_j : L|_{U_j} \to U_j \times \mathbb{R}$. We can then define a local section s_j on U_j by setting $s_j(x) \coloneqq \varphi_j^{-1}(x, 1),\ x \in U_j$ with the property $\varphi_j(\lambda s_j(x)) = (x, \lambda) \in U_j \times \mathbb{R}$. We can then define on $U_j \cap U_k$ a transition function $g_{jk} : U_j \cap U_k \to \mathbb{R}^\times = \mathrm{GL}_1(\mathbb{R})$ for sections via $s_j = g_{kj}s_k$ satisfying the following cocycle condition:

1. $g_{jj} = 1$ on U_j
2. $g_{jk}g_{kj} = 1$ on $U_j \cap U_k$
3. $g_{jk}g_{kl}g_{lj} = 1$ on $U_{jkl} = U_j \cap U_k \cap U_l$

Proposition 2.4.36. Let L be a line bundle with an open cover $\{U_j\}_{j\in J}$ of M with local trivialisations $\varphi_j : L|_{U_j} \to U_j \times \mathbb{R}$. Define $\{s_j\}_{j\in J}$ and $\{g_{jk}\}_{j,k\in J}$ as in Example 2.4.35. For each $s \in \Omega^0(M, L)$ there is a collection $\{f_j\}_{j\in J}$ where each $f_j \in C^\infty(U_j)$ satisfies:

1. $s|_{U_j} = f_j s_j,\ j \in J$
2. $f_k = g_{kj}f_j,\ k, j \in J$

Conversely every collection $\{f_j\}_{j\in J},\ f_j \in C^\infty(U_j)$ satisfying $f_k = g_{kj}f_j,\ k, j \in J$ yields a global section $s \in \Omega^0(M, L)$ with $s|_{U_j} = f_j s_j, j \in J$.

Definition 2.4.37. Let $\pi : E \to N$ be a fibre bundle and $\phi : M \to N$ be a smooth map. A section of E along ϕ is a smooth map $s : M \to E$ such that $s(x) \in E_{\phi(x)}$. We denote the space of sections by $\Omega^0_\phi(M, E)$. For $U \subseteq M$, local sections along ϕ are smooth maps $s : U \to E|_U$. We denote the space of local sections by $\Omega^0_\phi(U, E)$.

Definition 2.4.38. Let $\pi : E \to M$ be a vector bundle of rank k and let $U \subseteq M$ be open. A local frame for E over U is a collection $(e_1, ..., e_k)$ of sections $e_i \in \Omega^0(U, E)$ such that $(e_1(x), ..., e_k(x))$ is a basis of the space E_x for each $x \in U$. If $U = M$ the local frame is called a global frame.

Remark 2.4.39. For a line bundle a frame is a nowhere vanishing section.

Remark 2.4.40. If $(e_1, ..., e_k)$ is a local frame for E over U then any section $s : U \to E$ can be written as $s = a^i e_i$, for functions $a^i : U \to \mathbb{R}$. Giving a local frame is the same as giving a local trivialisation.

Finally we define a very important section of the vector bundle $(E \otimes E)^*$, a Riemannian metric.

Definition 2.4.41. Let $\pi : E \to M$ be a vector bundle. Recall from the Metatheorem that $(E \otimes E)$ is another vector bundle with the structure inherited from E. A Riemannian metric of E is a section $m \in \Omega^0(M, (E \otimes E)^*)$ with the property that for all $x \in M$, the element $m_x \in E_x^* \otimes E_x^* \cong (E \otimes E)^*|_x$ is an inner product on the vector space E_x, i.e. a bilinear, symmetric, positive definite form. We call the pair (E, m) a Riemannian vector bundle.

2.4.6 Hermitian and Holomorphic Vector Bundles

We first define complex vector and line bundles and then specify to Hermitian vector bundles and holomorphic vector bundles.

Definition 2.4.42. A complex vector bundle is a vector bundle whose fibres are complex vector spaces. In other words we can define a rank k complex vector bundle by taking Definition 2.4.11 and replacing every $\mathbb{R}$ with $\mathbb{C}$. Thus the transition functions take value in $\mathrm{GL}_k(\mathbb{C})$ matrices.

Remark 2.4.43. A rank k complex vector bundle is a rank $2k$ real vector bundle but not conversely, because of the choice of transition maps. Indeed $\mathrm{GL}_k(\mathbb{C})$ is a subgroup of $\mathrm{GL}_{2k}(\mathbb{R})$.

Remark 2.4.44. Let E be a complex vector bundle. The conjugate vector bundle $\overline{E}$ has the same complex vector bundle structure as E only that complex numbers will act by their complex conjugate on $\overline{E}$. This construction is allowed by the Metathorem since it is basis free.

Just as we defined smooth family of Riemannian metrics, we will define Hermitian metrics.

Definition 2.4.45. Let $\pi : E \to M$ be a complex vector bundle. A Hermitian metric of E is a section $h \in \Omega^0(M, (E \otimes \overline{E})^*)$ with the property that for all $x \in M$, the element $h_x \in E_x^* \otimes \overline{E}_x^* = (E \otimes \overline{E})^*|_x$ is a Hermitian inner product on the vector space E_x. In other words a Hermitian metric is a smoothly varying positive definite sesquilinear Hermitian form on each fibre. We call the pair (E, h) a Hermitian vector bundle.

Example 2.4.46. In the case of the trivial complex line bundle $L = M \times \mathbb{C}$ we define a Hermitian metric h_0 by setting:

$$h_0((a, z), (a, w)) \coloneqq z\overline{w}, \ z, w \in \mathbb{C}, \ a \in M.$$

We call h_0 the constant Hermitian metric, which is sensible since h_0 is not varying over L. One can prove that any other Hermitian metric h on L is given by a smooth function $\widetilde{h} : M \to \mathbb{R}_{>0}$ by

$$h((a, z), (a, w)) \coloneqq \widetilde{h}(a) \ h_0((a, z), (a, w)) = \widetilde{h}(a)z\overline{w}, \ \forall a \in M.$$

Proposition 2.4.47. Every complex vector bundle admits a Hermitian metric.

Remark 2.4.48. The proof of Proposition 2.4.47 is given by the application of partition of unity, which can be found in [1], Definition 3.12.

Example 2.4.49. Let (L, h) be a Hermitian line bundle. We can associate to it the circle bundle $L^1 \to M$, where

$$L^1 \coloneqq \{l \in L \mid h(l, l) = 1\}$$

This is a principal bundle with the circle group $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ as its structure group. Conversely if $P \to M$ is a principal S^1-bundle and $\rho : S^1 \to \mathbb{C}^\times = \mathrm{GL}_1(\mathbb{C})$ is the tautological representation $\rho(z) : \mathbb{C} \to \mathbb{C},\ w \mapsto zw$, then the associated vector bundle $L = P \times_\rho \mathbb{C}$ is a line bundle where S^1 acts by scalar multiplication. The Hermitian metric h on L is then given by:

$$h([x, z], [y, w]) \coloneqq z\overline{w},\ x, y \in P,\ z, w \in \mathbb{C}$$

We will see later on what it means for a connection to be compatible with a Hermitian metric. This will be central in the prequantisation process. Finally we define holomorphic vector bundles.

Definition 2.4.50. Let $U \subseteq \mathbb{C}^n$ be an open subset. A complex valued function $f : U \to \mathbb{C}$ is holomorphic if it satisfies the Cauchy–Riemann equations:

$$\frac{\partial f}{\partial \bar{z}_j} = 0,\ \forall j \in \{1, ..., n\}$$

where $z_j = x_j + iy_j$ denotes the standard coordinates on $\mathbb{C} \subseteq \mathbb{C}^n$ and $\frac{\partial}{\partial \bar{z}_j} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$. We denote the space of holomorphic functions on U with $\mathcal{O}_U$.

Definition 2.4.51. Let $U \subseteq \mathbb{C}^n$ be an open subset. A function $f : U \to \mathbb{C}^m$ is holomorphic if all of its components are holomorphic.

Definition 2.4.52. A complex manifold M of complex dimension n is a smooth manifold of dimension $2n$ such that there exists an atlas of local charts

$$\sigma_a : U_a \to O_a \subseteq \mathbb{C}^n,\ a \in A$$

where $\{U_a \mid a \in A\}$ is an open cover of M, $O_a \subseteq \mathbb{C}^n$ is open, and the transition maps:

$$\sigma_b \circ \sigma_a^{-1} : \sigma_a(U_a \cap U_b) \to \sigma_b(U_a \cap U_b)$$

are biholomorphic, i.e. $\sigma_b \circ \sigma_a^{-1}$ and $\sigma_a \circ \sigma_b^{-1}$ are holomorphic.

Remark 2.4.53. A complex manifold of dimension k is a real manifold of dimension $2k$ but not conversely since holomorphic maps are smooth but not conversely.

Definition 2.4.54. Let $\phi : M \to N$ a continuous map between two complex manifolds. Let $x \in M$ be a point with U_a a neighbourhood of x and U_b a neighbourhood of $\phi(x)$. Let $\sigma : U_a \to O_a \subseteq \mathbb{C}^{\dim_{\mathbb{C}}(M)}$ be any chart on M and $\tau : U_b \to O_b \subseteq \mathbb{C}^{\dim_{\mathbb{C}}(N)}$ be any chart on N. Then ϕ is called a holomorphic map if the composition:

$$\tau \circ \phi \circ \sigma^{-1} : \sigma(U_a \cap \phi^{-1}(U_b)) \to \tau(\phi(U_a) \cap U_b)$$

is holomorphic.

Definition 2.4.55. A complex vector bundle $\pi : E \to M$ of rank n over a complex manifold M is holomorphic if E is a complex manifold, $\pi : E \to M$ is a holomorphic map and there exists a system of local trivialisations $(U_a, \varphi_a)_{a \in A}$ where the maps:

$$\varphi_a : E|_{U_a} \to U_a \times \mathbb{C}^n$$

are biholomorphic.

Similarly to the compatibility of a connection with a Hermitian metric, we will see that we also need the connection to be compatible with the holomorphic structure on E.

2.5 Differential Forms

We first define tensor products and exterior algebras for vector spaces which will generalise to vector bundles with the Metatheorem. We then define differential r-forms on M which are sections of the bundle whose total space is the exterior algebra of the cotangent bundle of M. We then define the exterior derivative which is an operator acting on differential forms. We finally define bundle valued forms which generalise differential forms.

2.5.1 Tensor Product

For the following let V and W be two vector spaces on a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

Definition 2.5.1. Let $\mathrm{Free}(V)$ denote the vector space which has V as a basis. Therefore $\mathrm{Free}(V, W)$ will denote the vector space which has as basis all the elements of the form (v, w) where $v \in V$ and $w \in W$. Let $R(V, W)$ denote the linear subspace of $\mathrm{Free}(V, W)$ generated by the set of all elements of the form

$$\begin{aligned}
(v_1 + v_2, w) - (v_1, w) - (v_2, w),\ v_1, v_2 \in V,\ w \in W \\
(v, w_1 + w_2) - (v, w_1) - (v, w_2),\ v \in V,\ w_1, w_2 \in W \\
c(v, w) - (cv, w),\ v \in V,\ w \in W,\ c \in \mathbb{K} \\
c(v, w) - (v, cw),\ v \in V,\ w \in W,\ c \in \mathbb{K}
\end{aligned}$$

Definition 2.5.2. Let V and W be two vector spaces. Their tensor product $V \otimes W$ is the vector space defined by $\mathrm{Free}(V, W)/R(V, W)$. The coset in $V \otimes W$ containing (v, w) is denoted by $v \otimes w$. The elements contained in $V \otimes W$ are therefore of the form

$$\begin{aligned}
(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w,\ v_1, v_2 \in V,\ w \in W \\
v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2,\ v \in V,\ w_1, w_2 \in W \\
c(v \otimes w) = (cv) \otimes w,\ v \in V,\ w \in W,\ c \in \mathbb{K} \\
c(v \otimes w) = v \otimes (cw),\ v \in V,\ w \in W,\ c \in \mathbb{K}
\end{aligned}$$

Lemma 2.5.3. (Universal Property) Let V, W and U be vector spaces and let $T : V \times W \to V \otimes W$ be the natural bilinear map that sends $(v, w) \mapsto v \otimes w$. Suppose $B : V \times W \to U$ is a bilinear map. Then there is a unique linear map $L : V \otimes W \to U$ such that $B(v, w) = (L \circ T)(v, w)$ for all $v \in V$ and $w \in W$. Moreover this property uniquely characterizes $V \otimes W$ (up to a canonical isomorphism).

Corollary 2.5.4. Let $V^* = \mathrm{Hom}(V, \mathbb{R})$ denote the dual space. Then there is a natural isomorphism between $\mathrm{Hom}(V, W)$ and $V^* \otimes W$.

Proof. Define $B : V^* \otimes W \to \mathrm{Hom}(V, W)$ by $B(\lambda, w)(v) := \lambda(v) \cdot w$. We then have a unique linear map $L : V \otimes W \to \mathrm{Hom}(V, W)$ by the universal property. This map is an isomorphism because the inverse of L is given by

$L^{-1} : \mathrm{Hom}(V, W) \to V^* \otimes W,\ h \mapsto \sum_i e^i \otimes h(e_i)$, where e_i is any basis of V and e^i is the dual basis of V^*. $\square$

Example 2.5.5. Let $\pi : E \to M$ be a vector bundle. With Corollary 2.5.4, we can write $L \otimes L^* \cong \mathrm{End}(L)$. Thus, it follows from the Metatheorem that $\mathrm{End}(L)$ defines a vector bundle as well.

Corollary 2.5.6. If $(e_1, ..., e_{\dim(V)})$ is a basis of V and $(e'_1, ..., e'_{\dim(W)})$ is a basis of W then $e_i \otimes e'_j$ is a basis for $V \otimes W$. Thus $\dim(V \otimes W) = \dim(V) \cdot \dim(W)$

Proof. It is clear that any element of $V \otimes W$ can be written as a linear combination of $e_i \otimes e'_j$, because of the bilinearity of the map T defined in Lemma 2.5.3.
For the linear independence we use the same map L as in the proof of Corollary 2.5.4 where we define $\lambda := e^j$, the jth element of the dual basis of V^*. Suppose that the linear combination $\sum_{k,l} \alpha_{kl} e_k \otimes e'_l,\ \alpha_{kl} \in \mathbb{K}$ equals zero. Then by applying L we get the equation $0 = \sum_{kl} \alpha_{kl} e^j(e_k) e'_l = \sum_l \alpha_{jl} e'_l$. The e'_l are linearly independent since they form a basis of W and therefore $\alpha_{kl} = 0,\ \forall k, l$. $\square$

Corollary 2.5.7. Let V, W and U be vector spaces then there are natural isomorphisms $V \otimes W \cong W \otimes V$ and $(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$.

Definition 2.5.8. Let r and s be non-negative integers. A tensor of type (r, s) is an element of

$$\underbrace{V \otimes ... \otimes V}_{r-\text{times}} \otimes \underbrace{V^* \otimes ... \otimes V^*}_{s-\text{times}}$$

We denote this vector space by $T^{r,s}(V)$.

Note that this definition makes sense thanks to Corollary 2.5.7. Additionally $T^{r,s}(V)$ is obviously $(\dim(V))^{r+s}$-dimensional by Corollary 2.5.6.

Remark 2.5.9. Let $\pi : E_i \to M$ be vector bundles for $i \in \{1, 2, 3\}$. Then the above Lemma 2.5.3, Corollaries 2.5.4, 2.5.6, 2.5.7 and Definitions 2.5.2, 2.5.8 can be generalized for vector bundles by replacing V, W, U with E_1, E_2, E_3, thanks to the Metatheorem 2.4.16.

Remark 2.5.10. The direct sum over all the tensors of type (r, s) defines an algebra where the product is naturally the tensor product $\otimes$.

2.5.2 Exterior Algebra

We now define exterior algebras which are, for our purpose, much more interesting than tensor products, because of differential forms.

Definition 2.5.11. Let $(R, +, \cdot)$ be a ring with additive subgroup $(R, +)$. A subset I is a left ideal of R if it satisfies the conditions:

1. $(I, +) \subseteq (R, +)$ is a subgroup

2. $r \cdot i \in I$, for $r \in R$ and $i \in I$.

A two sided ideal is a left ideal which is at the same time a right ideal, i.e. the second condition of the definition above is also satisfied with the product from the right.

Proposition 2.5.12. If $(R, +, \cdot)$ is a ring and I is a two sided ideal, then R/I can be endowed with a ring structure.

Definition 2.5.13. Let $T^+(V)$ denote the subalgebra given by $T^+(V) := \bigoplus_{r\geq 0} T^{r,0}(V)$. Let $I(V)$ denote the two-sided ideal in $T^+(V)$ generated by all elements of the form $v \otimes v$ for $v \in V$. The exterior algebra is defined as the quotient algebra $\Lambda(V) := T^+(V)/I(V)$. We denote the image of $v_1 \otimes ... \otimes v_r$ in $\Lambda(V)$ by $v_1 \wedge ... \wedge v_r$ and call $\wedge$ the product on the quotient, which we call the wedge product.

Remark 2.5.14. There is a canonical vector space isomorphism:

$$\Lambda^r(V) \cong T^{r,0}(v)/I^r(V)$$

where $I^r(V) := T^{r,0}(V) \cap I(V)$.

Proposition 2.5.15. The wedge product satisfies the following properties:

1. Let $r, s > 0$. If $v \in \Lambda^r(V)$ and $w \in \Lambda^s(V)$ then $v \wedge w \in \Lambda^{r+s}(V)$ and $v \wedge w = (-1)^{r \cdot s} w \wedge v$

2. If ρ is a permutation of the set $\{1, ..., r\}$ and $\{v_i\}_{i\in\{1,..,r\}}$ in V then $v_{\rho(1)} \wedge ... \wedge v_{\rho(r)} = \mathrm{sgn}(\rho) v_1 \wedge ... \wedge v_r$.

Definition 2.5.16. Let $\mathrm{Alt}_r(V, W)$ denote the space of alternating r-linear maps, i.e. multilinear maps $A : V \times ... \times V \to W$ which satisfy $A(v_1, v_2, ..., v_r) = 0$ if there exist $i \neq k$ such that $v_i = v_k$.

Remark 2.5.17. Analog to Definition 2.5.16, one could define alternating maps as maps $A\colon V^{\otimes r} \to \mathbb{R}$ which vanish on $I^r(V)$.

We will denote $\mathrm{Alt}_r(V)$ for $\mathrm{Alt}_r(V, \mathbb{R})$.

Proposition 2.5.18. There is a canonical isomorphism between $\Lambda^r(V^*)$ and $\mathrm{Alt_r(V)}$.

Remark 2.5.19. Analogously to the universal property for tensor products one can prove that there is a universal property for the exterior algebra by replacing any tensor product $\otimes$ by a wedge product $\wedge$ and any bilinear map by an alternating map in Lemma 2.5.3.

Proposition 2.5.20. Let V be a vector space of dimension k with basis $(e_1, ..., e_k)$. Then $(e_{i_1} \wedge ... \wedge e_{i_r} \mid 1 < i_1 < .... < i_r < k)$ is a basis of $\Lambda^r(V)$ and $\Lambda^r(V) = 0$ for rank $r > k$. Thus $\dim(\Lambda^r(V)) = \binom{k}{r}$.

2.5.3 Differential forms

Differential forms are extremely important because they can be used to make sense of integration on oriented compact manifolds. Moreover we will define symplectic manifolds which are manifolds equipped with a symplectic form.

Definition 2.5.21. Let M be a smooth manifold of dimension n and let $0 \leq r \leq n$. A differential r-form is a section of the bundle $\Lambda^r(T^*M) \to M$. We denote the space of differential r-forms by $\Omega^r(M) \coloneqq \Omega^0(M, \Lambda^r(T^*M))$. If $U \subseteq M$ is an open subset of M we write $\Omega^r(U) = \Omega^0(U, \Lambda^r(T^*M))$. We denote the space of differential forms by $\Omega(M) = \bigoplus_{0 \leq r \leq n} \Omega^r(M)$.

Remark 2.5.22. Differential 0-forms $\Omega^0(M)$ are smooth functions.

Theorem 2.5.23. *(The Differential Form Criterion)*
Let $U \subseteq M$ be an open set. Then there is a canonical identification between $\Omega^r(U)$ and alternating $C^\infty(U)$-multilinear functions:

$$\underbrace{\Omega^0(U,TM) \times ... \times \Omega^0(U,TM)}_{r-times} \to C^\infty(U)$$

.

Remark 2.5.24. This is a global version of Proposition 2.5.18. The isomorphism from Theorem 2.5.23 is given by the map

$$\begin{aligned} \phi \colon \Omega^r(U) &\to \left(\Omega^0(U,TM)^{\otimes r}\right)^* \otimes C^\infty(U), \\ \omega &\mapsto \big((X_1, ..., X_r) \mapsto \omega(X_1, ..., X_r)\big). \end{aligned}$$

Recall that $\left(\Omega^0(U,TM)^{\otimes r}\right)^* \otimes C^\infty(U) \cong \mathrm{Hom}(\Omega^0(U,TM)^{\otimes r}, C^\infty(U))$ from Corollary 2.5.4.

Definition 2.5.25. If $\omega \in \Omega^r(M)$ and $\theta \in \Omega^s(M)$ then the wedge product is the differential form $\omega \wedge \theta \in \Omega^{r+s}(M)$ defined pointwise by $(\omega \wedge \theta)(x) \coloneqq \omega(x) \wedge \theta(x)$, where on the right hand side we use the wedge product from Definition 2.5.13.

Remark 2.5.26. We note that Definition 2.5.25 implies that the wedge product for differential forms inherits the properties from Proposition 2.5.15.

Proposition 2.5.27. Let $\sigma : U \to O$ be a chart on M. A local frame for $\Lambda^r(T^*M) \to M$ over U is given by $(dx_{i_1} \wedge .... \wedge dx_{i_r} \mid i_1 < ... < i_r)$ and we can locally write a differential r-form as $\omega = \sum_{i_1 < ... < i_r} \omega_{i_1 ... i_r} dx_{i_1} \wedge ... \wedge dx_{i_r}$ where $\omega_{i_1 ... i_r} \in C^\infty(U)$.

We now discuss pullback forms which are going to be central when discussing connections on principal G-bundles and on associated bundles.

Definition 2.5.28. Let $\phi : M \to N$ be a smooth map. Given $\omega \in \Omega^r(N)$ we define the pullback form $\phi^*\omega \in \Omega^r(M)$ by

$$\phi^*\omega_x(v_1, ..., v_r) \coloneqq \omega_{\phi(x)}(D\phi(x)[v_1], ..., D\phi(x)[v_r])$$

This defines a map $\phi^* : \Omega^r(N) \to \Omega^r(M)$ for all r and thus also a map $\phi^* : \Omega(N) \to \Omega(M)$.

Lemma 2.5.29. If $\phi : M \to N$ is a smooth map and $\omega, \theta \in \Omega(N)$ then

$$\phi^*(\omega \wedge \theta) = \phi^*(\omega) \wedge \phi^*(\theta),$$

i.e. ϕ^* is an algebra homomorphism.

We now give an axiomatic definition of the exterior derivative which gives us a way to derive the differential forms.

Definition 2.5.30. The exterior derivative is the unique $\mathbb{R}$-linear map $d\colon \Omega(M) \to \Omega(M)$ such that the following properties are satisfied

1. df is the differential of f, for smooth functions f.
2. $d^2 = 0$
3. $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{kl}(\alpha \wedge d\beta)$, for $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^l(M)$.

Remark 2.5.31. The exterior derivative d sends k-forms to $k+1$-forms. We note that this implies that the exterior derivative sends n-forms to zero because of the dimension formula from Proposition 2.5.20.

Definition 2.5.32. A differential form ω is said to be closed if $d\omega = 0$ and it is said to be exact if $\omega = d\theta$ for some θ another differential form. In other words closed forms are the kernel of the exterior derivative and exact forms are the image of the exterior derivative.

Note that it follows directly from Definition 2.5.30 that an exact form is also closed.

2.5.4 Bundle Valued Forms

We now generalise differential forms by letting them take values in a vector space or a vector bundle. We first define vector valued forms.

Definition 2.5.33. Let W be a vector space. A differential r-form on M with values in W, or a W-valued form, is a smooth section of the bundle $\Lambda^r(T^*M) \otimes W \to M$. We denote the space of W-valued r-forms by

$$\Omega^r(M, W) \coloneqq \Omega^0(M, \Lambda^r(T^*M) \otimes W).$$

Remark 2.5.34. This is indeed the good notion for a differential taking values in W. Let $\omega \in \Omega^r(M, W)$ and $x \in M$. Then we have $\omega_x \in \Lambda^r(T_x^*M) \otimes W \cong \Big(\Lambda^r(T_xM)\Big)^* \otimes W$. By Corollary 2.5.4 $\Big(\Lambda^r(T_xM)\Big)^* \otimes W$ is isomorphic to $\mathrm{Hom}(\Lambda^r(T_xM), W)$, hence $\omega_x \in \mathrm{Alt}_r(T_xM, W)$ by the universal property.
In a local trivialisation we can write ω as

$$\omega_x(v_1, ..., v_r) = \sum_{i=1}^{k} \omega_x^i(v_1, ..., v_r)e_i$$

where $v_1, ..., v_r \in T_xM$, $(e_1, ..., e_k)$ a basis of W and ω_x^i an alternating $\mathbb{R}$-linear map. The ω_x^i are differential r-forms on M and so we can alternatively write $\omega = \sum_i \omega_i \otimes e_i$.
Therefore, almost everything showed in Subsection 2.5.3 can be carried on to vector-valued forms. Theorem 2.5.36 is an example of this.

Example 2.5.35. The special case where $W = \mathbb{C}$ in Definition 2.5.33 is very important for complex geometry.

Theorem 2.5.36. *(The Vector Valued Form Criterion) Let W be a vector space. Then there is a canonical isomorphism between $\Omega^r(M, W)$ and alternating $C^\infty(M)$-multilinear functions*

$$\underbrace{\Omega^0(M, TM) \times ... \times \Omega^0(M, TM)}_{r-times} \to C^\infty(M, W).$$

Definition 2.5.37. Suppose W_1, W_2, Z are vector spaces and assume we have a bilinear map $\beta\colon W_1 \times W_2 \to Z$. Let $\omega \in \Omega^r(M, W_1)$, $\theta \in \Omega^s(M, W_2)$ and $v_1, ..., v_{r+s} \in T_xM$. Then we write:

$$(\omega_x \wedge_\beta \theta_x)(v_1, ..., v_{r+s}) \coloneqq \frac{1}{r!s!} \sum_{\rho \in \mathcal{G}_{r+s}} \operatorname{sgn}(\rho)\beta(\omega(v_{\rho(1)}, ..., v_{\rho(r)}), \theta(v_{\rho(r+1)}, ..., v_{\rho(r+s)})),$$

where $\mathcal{G}_{r+s} \subseteq \Sigma_{r+s}$ is the subgroup of (r, s)-shuffles within the symmetric group of permutations of its objects. An (r, s)-shuffle $\rho \in \mathcal{G}_{r+s}$ is a permutation such that $\rho(1) < ... < \rho(r)$ and $\rho(r + 1) < ... < \rho(r + s)$.

Example 2.5.38. Let $\mathfrak{g}$ be a Lie algebra. Then we can take the bilinear map β to be the Lie bracket $[\cdot, \cdot]$ of $\mathfrak{g}$. Given $\omega \in \Omega^r(M, \mathfrak{g})$ and $\theta \in \Omega^s(M, \mathfrak{g})$, we will use the notation $[\omega, \theta] \coloneqq \omega \wedge_{[\cdot,\cdot]} \theta$ and one can show that $[\omega, \theta] = (-1)^{rs+1}[\theta, \omega]$.

Now bundle valued forms can be defined analogously.

Definition 2.5.39. Let $\pi\colon E \to M$ be a vector bundle. A differential r-form on M with values in E, or E-valued form, is a section of the bundle $\Lambda^r(T^*M) \otimes E \to M$. We denote the space of E-valued forms by

$$\Omega^r(M, E) \coloneqq \Omega^0(M, \Lambda^r(T^*M) \otimes E).$$

Remark 2.5.40. Analogously to Remark 2.5.34, any bundle-valued form $\omega \in \Omega^r(M, E)$ can be viewed as an alternating $\mathbb{R}$-linear map

$$\omega_x\colon \underbrace{T_xM \times ... \times T_xM}_{r-\text{times}} \to E_x$$

If $(e_1, ..., e_k)$ is a local frame for E over an open set $U \subseteq M$ then any element $\omega \in \Omega^r(U, E)$ can be written as a sum $\omega = \sum_i \omega_i \otimes e_i$ where ω_i is a differential form on U.

We also get a bundle valued form criterion that we mention for clarity and the rest can be similarly adapted.

Theorem 2.5.41. *(The Bundle-valued Form Criterion) Let E be a vector bundle. Then there is a natural isomorphism between $\Omega^r(M, E)$ and alternating $C^\infty(M)$-multilinear functions*

$$\underbrace{\Omega^0(M, TM) \times ... \times \Omega^0(M, TM)}_{r-times} \to \Omega^0(M, E).$$

2.6 Symplectic Manifolds

As mentioned above we will describe our classical (Hamiltonian) systems via symplectic manifolds.

Let V be an m-dimensional vector space over $\mathbb{R}$ and let $\omega : V \times V \to \mathbb{R}$ be a bilinear map.

Definition 2.6.1. The map $\widetilde{\omega} : V \to V^*$ is defined by $\widetilde{\omega}\big(u\big)(v) = \omega(u, v)$ for $u, v \in V$.

Remark 2.6.2. It is clear that $\widetilde{\omega}$ is linear.

Definition 2.6.3. A skew-symmetric bilinear map ω is non-degenerate if $\widetilde{\omega}$ from Definition 2.6.1 is an isomorphism. In other words ω is non-degenerate if $\omega(u, v) = 0$, for all $v \in V$ implies that $u = 0$.

Definition 2.6.4. A symplectic manifold is a pair (M, ω) where the symplectic form ω is a closed non-degenerate differential 2-form on M.

Example 2.6.5. The symplectic manifold (M, ω) that we will study in section 5.3 will describe a 2-dimensional Hamiltonian system. We write $M = T^*\mathbb{R}$ with global canonical coordinates q and p. The position coordinate is q and the momentum coordinate is p varying on the cotangent fibres. Then we claim that the 2-form $\omega := dq \wedge dp$ defines a symplectic form on M. It is closed, since it is exact. Indeed the 1-form $\alpha = -pdq$ satisfies $d\alpha = -dp \wedge dq = dq \wedge dp = d\omega$. For the non-degeneracy it is particularly instructive to look at the matrix of the bilinear form ω. We have

$$\omega = dq \wedge dp = 0 \cdot dq \otimes dq + dq \otimes dp - dp \otimes dq + 0 \cdot dp \otimes dp$$

that means that the matrix of ω in this frame is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Since the determinant of this matrix equals 1 it follows that ω is non-degenerate.

Remark 2.6.6. A differential 2-form is skew symmetric by definition, in particular the symplectic form ω fulfils $\omega(x, y) = -\omega(y, x),\ x, y \in M$. Moreover, the symplectic form is non-degenerate and therefore must have full rank. Since a skew symmetric matrix is not invertible in odd dimensions, we understand that a symplectic manifold needs to have even dimension.

Theorem 2.6.7. *(Darboux's Theorem) Every point $a \in M$ of a symplectic manifold has an open neighbourhood $U \subseteq M$ and a chart*

$$\sigma = (q_1, ..., q_n, p_1, ..., p_n) \colon U \to V \subseteq \mathbb{R}^n \oplus \mathbb{R}^n,$$

such that $\omega|_U = \sum_j dq_j \wedge dp_j$. The q_j, p_k are called Darboux coordinates.

Remark 2.6.8. We understand from Darboux's theorem that every symplectic manifold is locally isomorphic to $T^*\mathbb{R}^n$. This motivates why we are considering $T^*\mathbb{R}$ in Example 2.6.5. We will continue the discussion of this flat case later on in Hamiltonian mechanics as well as in the quantisation process.

2.7 Connections

As mentioned above we are going to construct the Hilbert phase space of the quantum system out of sections on vector bundles. One could then ask how the derivative (which is central for physical purposes) is defined for sections on vector bundles. The answer to that question is given by connections. There are many definitions of connections. We will only cite a few of them. We will also discuss connections on principal G-bundles since, as explained above, they are more general than vector bundles and we can therefore gain some insight about connections on vector bundles thanks to such an approach.

2.7.1 Covariant Derivative

We first define a connection as a first order differential operator. We then define a Koszul connection, which is the most familiar way of defining a connection. These definitions are only valid for vector bundles.

Definition 2.7.1. Let $\pi : E \to M$ be a vector bundle. A connection is a first order differential operator $\nabla : \Omega^0(M, E) \to \Omega^1(M, E)$ such that:

$$\nabla(fs) = df \otimes s + f\nabla s, \text{ where } f \in C^\infty(M) \text{ and } s \in \Omega^0(M, E).$$

Definition 2.7.2. Let $\pi : E \to M$ be a vector bundle. A covariant derivative operator in E is a map $\nabla : \Omega^0(M, TM) \times \Omega^0(M, E) \to \Omega^0(M, E)$ written $(X, s) \mapsto \nabla_X s$ which satisfies the following for any $X, Y \in \Omega^0(M, TM)$, $s_1, s_2 \in \Omega^0(M, E)$ and $f \in C^\infty(M)$:

1. $\nabla_{X+fY} s = \nabla_X s + f\nabla_Y s$ (linear on vector fields);
2. $\nabla_X(s_1 + s_2) = \nabla_X s_1 + \nabla_X s_2$ (additive on section);
3. $\nabla_X(fs) = \big(X.f\big)s + f\nabla_X s$ (Leibniz rule).

In words, the covariant derivative operator in E associates to any vector field $X \in \Omega^0(M, TM)$ a linear map $\Omega^0(M, E) \to \Omega^0(M, E)$ satisfying Leibniz. We call ∇_X the covariant derivative operator along X.

Theorem 2.7.3. *Every vector bundle admits a connection.*

Remark 2.7.4. The proof of 2.7.3 is given by a partition of unity argument.

We now specialise to line bundles which are particularly important for geometric quantisation. First we look at trivial line bundles.

Remark 2.7.5. For the following lemma, it will be important to understand that $\Omega^1(M, \mathrm{End}(L)) \cong \Omega^1(M, \mathbb{C})$ for a trivial line bundle $\pi\colon L \to M$. From Example 2.5.5, we know that $\mathrm{End}(L)$ defines a line bundle. It is in addition trivialisable since we can find a nowhere vanishing global section (see Proposition 2.4.34), for example $s\colon M \to \mathrm{End}(L)$, $x \mapsto \mathrm{id}_{L_x}$. Hence, we can write $\Omega^1(M, \mathrm{End}(L))$ as the space of $\mathbb{C}$-valued 1 forms $\Omega^1(M, \mathbb{C})$.

Lemma 2.7.6. Each connection ∇ on $L = M \times \mathbb{C} \to M$ is of the form

$$\nabla = d - \beta \tag{2.2}$$

with d the exterior derivative acting on a coefficient function and $\beta \in \Omega^1(M, \mathrm{End}(L)) \cong \Omega^1(M, \mathbb{C})$ a suitable complex differential form. Conversely for each $\beta \in \Omega^1(M, \mathbb{C})$ the above formula defines a connection on L.

Proof. Let us first check that $\nabla := d - \beta$ is a connection for each $\beta \in \Omega^1(M, \mathbb{C})$. Let $f, g \in C^\infty(M)$ be smooth functions on M and $s = gs_1 \in \Omega^0(M, L)$ be a section, where s_1 is the constant section. We calculate

$$\begin{aligned}\nabla(fs) &= (d - \beta)(fs)\\ &= df \otimes gs_1 + fdg \otimes s_1 - \beta fgs_1\\ &= df \otimes gs_1 + f(d - \beta)gs_1\\ &= df \otimes s + f\nabla s.\end{aligned}$$

This shows that ∇ is a connection since it fulfils the condition from Definition 2.7.1.

Conversely, let ∇ be an arbitrary connection on L. We know that ∇ fulfils the condition from Definition 2.7.1. Therefore

$$\nabla(gs_1) = dg \otimes s_1 + g\nabla s_1.$$

Since $\nabla s_1 \in \Omega^1(M, M \times \mathbb{C}) \cong \Omega^1(M, \mathbb{C})$, we can define the 1-form $\beta \in \Omega^1(M, \mathbb{C})$ such that $-\beta \otimes s_1 = \nabla s_1$. This is indeed a sensible definition since we know that a global nowhere-vanishing section is equivalent to a global frame on a line bundle. It follows that

$$\nabla s = dg \otimes s_1 + g\nabla s_1 = (d - \beta)s.$$

$\square$

Remark 2.7.7. Let us unpack Lemma 2.7.6. We know from Example 2.4.33 that any section of a trivial line bundle is of the form fs_1 with $f \in C^\infty(M)$ and s_1 the constant section. Combining Definition 2.2.38 and Remark 2.2.49, we know that $< df, X >= X.f$ is well defined and gives another function in $C^\infty(M)$. Finally $< \beta, X >$ means that we are regarding $\beta \in \Omega^1(M, \mathbb{C})$ as being the alternating $C^\infty(M)$-multilinear map eating vector fields on M and outputting functions in $C^\infty(M, \mathbb{C})$ as showed in Theorem 2.5.36. Therefore letting (2.2) act on a section fs_1 along a vector field X yields:

$$\nabla_X(s) = (X.f + \beta|_X f)(s_1),\ s = fs_1|_U,\ f \in C^\infty(U),\ U \subseteq M.$$

We now want to describe connections on any line bundle locally with Lemma 2.7.6.

Proposition 2.7.8. Let $\pi : L \to M$ be a possibly non-trivial line bundle and $(U_j, \varphi_j)_{j\in J}$ be a local trivialisation system with corresponding transition functions $g_{jk} \in C^\infty(U_j \cap U_k, \mathbb{C}^\times)$ (recall Proposition 2.4.36). Any connection ∇ determines uniquely a collection $\{\alpha_j\}_{j\in J}$ of 1-forms $\Omega^1(U_j, \mathbb{C})$ satisfying:

$$\alpha_k - \alpha_j = dg_{jk}g_{jk}^{-1}$$

Conversely any collection $\{\alpha_j\}_{j\in J}$ satisfying the condition above induces a connection on L with restriction on U_j given by:

$$\nabla_X(fs_j) = (X.f + \alpha_j|_X f)(s_j),\ s_j = \varphi_j^{-1}(x, 1),\ \forall x \in U_j.$$

Thus the information of the connection is encoded in the α_j on the local trivialisations. The α_j are called local connection forms.

2.7.2 Principal Bundle Connection

We will be able to induce a connection on the associated bundle of a principal bundle, see Subsection 2.7.3.

Definition 2.7.9. Let $\pi : E \to M$ be a fibre bundle over a smooth manifold with fibre F. A preconnection on E is a distribution $\mathcal{H}$ on E with the additional property that for every $p \in E$ the map $D\pi(p)|_{\mathcal{H}_p} : \mathcal{H}_p \to T_{\pi(p)}M$ is a linear isomorphism.

Theorem 2.7.10. *Every fibre bundle admits a preconnection.*

Definition 2.7.11. Let $\pi : E \to M$ be a fibre bundle over a smooth manifold with fibre F. Then the vertical bundle $VE \subseteq TE$ is the subbundle of the tangent bundle of E whose fibre over $p \in E$ is $\ker D\pi(p)$, where $D\pi(p) : T_pE \to T_{\pi(p)}M$. A tangent vector ζ at p is said to be vertical if $\zeta \in V_pE$, i.e. if $D\pi(p)[\zeta] = 0$.

Remark 2.7.12. We then understand that a preconnection is a distribution on E which is complementary to the vertical bundle VE, that is $TE = \mathcal{H} \oplus VE$. This is the geometrical interpretation of a preconnection.

Definition 2.7.13. Let $\pi : E \to M$ be a fibre bundle with preconnection $\mathcal{H}$. The splitting $TE = \mathcal{H} \oplus VE$ from Remark 2.7.12 allows us to write any vector $\zeta \in TE$ uniquely as $\zeta = \zeta^H + \zeta^V$ where for $\zeta \in T_pE$ we have $\zeta^H \in \mathcal{H}_p$ and $\zeta^V \in V_pE$. We call ζ^H the horizontal component and ζ^V the vertical component of ζ. A vector is horizontal if $\zeta^V = 0$ and vertical if $\zeta^H = 0$.

Definition 2.7.14. Let $x \in M$, $p \in E_x$ and $v \in T_xM$. Then the horizontal lift of v at p is the unique vector $\bar{v} \in \mathcal{H}_p$ such that $D\pi(p)[\bar{v}] = v$.

Definition 2.7.15. Let $X \in \Omega^0(M, TM)$ be a vector field. Then the horizontal lift of X is the unique vector field $\bar{X} \in \Omega^0(E, TE)$ such that $\bar{X}(p) \in T_pE$ is the horizontal lift of $X(\pi(p)) \in T_{\pi(p)}M$ at p for each $p \in E$.

Definition 2.7.16. Let $\pi\colon E \to N$ be a fibre bundle and let $\mathcal{H}$ be a preconnection on E. Suppose $\phi : M \to N$ is a smooth map and let $s \in \Omega^0_\phi(M, E)$ be a section along ϕ. We say that s is horizontal along ϕ if

$$Ds(x)[T_xM] \subseteq \mathcal{H}_{s(x)},\ \forall x \in M.$$

Remark 2.7.17. If ϕ from Definition 2.7.16 is the identity map, we say that the section $s \in \Omega^0(M, E)$ is horizontal.

Example 2.7.18. Take M to be the interval (a, b) and $\phi = \gamma\colon (a, b) \to N$ to be a smooth curve in N. Thus a section $c \in \Omega^0_\gamma(M, E)$ is a smooth curve in E such that $c(t) \in E_{\gamma(t)}$ for all $t \in (a, b)$. Moreover c is horizontal along γ if $c'(t) \in \mathcal{H}_{c(t)}, \forall t \in (a, b)$.

Let us now define a connection on a principal G-bundle (π, P, M, G). Let $\mu : P \times G \to G$ be the right action of G on P. We will denote by $r_a : P \to P$ the right G-action $r_a(p) \coloneqq \mu(p, a) = p \cdot a,\ p \in P,\ a \in G$.

Definition 2.7.19. A G-connection on P is a preconnection $\mathcal{H}$ which satisfies

$$Dr_a(p)[\mathcal{H}_p] = \mathcal{H}_{p\cdot a},\ \forall p \in P,\ a \in G. \tag{2.3}$$

Remark 2.7.20. Given a left action μ as in Definition 2.4.5 with Lie group $G = \mathrm{GL}_k(\mathbb{R})$, we can define, analogously to Definition 2.7.19, a connection $\mathcal{H}$ on a vector bundle $\pi\colon E \to M$ as a preconnection with the additional property that $D\mu_a(p)[\mathcal{H}_p] = \mathcal{H}_{a \cot p}$. Such a connection induces a covariant derivative and conversely a covariant derivative induces a connection (see [1], Theorem 31.10). We will from now on use these two terms interchangeably. Additionally, one can show that $s \in \Omega^0(M, E)$ is horizontal if and only if $\nabla_X(s) = 0$ for all $X \in \Omega^0(M, TM)$.

We will now give a definition of connection forms. These are 1-forms which give a more applicable definition of connections on principal G-bundles.

Definition 2.7.21. Let G be a Lie Group with Lie algebra $\mathfrak{g}$ and let P be a principal G-bundle. For each $v \in \mathfrak{g}$ we can define the fundamental vector field ξ_v on P via

$$\xi_v(p) \coloneqq \frac{d}{dt}\Big|_{t=0} p \cdot \exp(tv) \in T_pP,$$

where $\exp\colon \mathfrak{g} \to G$ is the exponential map (see [1] Section 10).

Remark 2.7.22. The fundamental vector field is indeed well defined since for $p \in P$, the curve $\gamma_p(t) \coloneqq p \cdot \exp(tv)$ is a curve in P with initial point $\gamma_p(0) = p \cdot e = p$. Thus $\gamma_p'(0)$ belongs to T_pP.

Proposition 2.7.23. Define the map $\eta_p : G \to P$, via $\eta_p(a) \coloneqq p \cdot a,\ p \in P$, then we can write the fundamental vector field via $D\eta_p(e)[v] = \xi_v(p)$.

Proof. Define the smooth curve γ_p as in Remark 2.7.22. Then we can write:

$$D\eta_p(e)[v] = \frac{d}{dt}\Big|_{t=0} \eta_p(\exp(tv)) = \frac{d}{dt}\Big|_{t=0} p \cdot \exp(tv) = \xi_v(p).$$

$\square$

Proposition 2.7.24. Let η_p be the map from Proposition 2.7.23. For any $p \in P$, the differential $D\eta_p(e)$ at e is an isomorphism $D\eta_p(e) : \mathfrak{g} \to V_p P$.

Definition 2.7.25. Let $\mathcal{H}$ be a G-connection on P. The connection form α of $\mathcal{H}$ is defined by

$$\alpha_p(\zeta) := D\eta_p(e)^{-1}[\zeta^V], \ p \in P, \ \zeta \in T_p P.$$

One can show that it is a $\mathfrak{g}$-valued 1-form $\alpha \in \Omega^1(P, \mathfrak{g})$.

Proposition 2.7.26. (Properties of the connection form) The connection form induced by the G-connection $\mathcal{H}$ from Definition 2.7.25 is smooth and has the following properties:

1. it is G-equivariant, i.e. $r_a^*(\alpha) = \mathrm{Ad}_{a^{-1}}(\alpha), \ \forall a \in G$
2. $\alpha(\xi_v) = v, \ \forall v \in \mathfrak{g}$.

Proposition 2.7.27. Any 1-form $\alpha \in \Omega^1(P, \mathfrak{g})$ which satisfies the properties listed in Proposition 2.7.26 defines a connection on P via $\mathcal{H}_p := \ker \alpha_p$.

Proof. We first note that $\ker(\alpha)$ is a subbundle of the tangent bundle TP since α is smooth. We know that any element of TP can be uniquely written in a vertical and horizontal part. Now from Proposition 2.7.24, we know that $\xi_\cdot(p) = D\eta_p(e)(\cdot)$ maps isomorphically $\mathfrak{g}$ to $V_p P$. The second property in Proposition 2.7.26 tells us that α maps ξ_v to $v \neq 0$. That means that $\ker(\alpha)$ automatically belongs to the horizontal part of TP. Thus it follows that $TP = \ker(\alpha) \oplus VP$ and $\mathcal{H}$ is a preconnection. Moreover, the first property in Proposition 2.7.26 tells us that α is G-equivariant, i.e. $Dr_a[\ker(\alpha)] \subseteq \ker(\alpha)$. Applying $Dr_{a^{-1}}$ on both sides yields:

$$\ker(\alpha) = Dr_{a^{-1}} \circ Dr_a[\ker \alpha] \subseteq Dr_{a^{-1}}[\ker(\alpha)]$$

and with the G-equivariance property $Dr_{a^{-1}}[\ker(\alpha)] \subseteq \ker \alpha$. Thus we have $Dr_a[\ker \alpha] = \ker \alpha$ which shows that $\ker \alpha$ is a connection. $\square$

Remark 2.7.28. We now understand that a G-connection on a principal G-bundle is equivalent to a connection form. These forms are called in physics global gauge potentials.

We would like now to discuss local gauge potentials which are induced by the global gauge potential (connection form).

Definition 2.7.29. Let $U \subseteq M$ be an open subset of M and $s \in \Omega^0(U, P)$ a section. The local gauge potential over U associated to the connection form α is defined as $A_s := s^*\alpha \in \Omega^1(U, \mathfrak{g})$.

Proposition 2.7.30. Let $s_i : U_i \to P,\ i \in \{1, 2\}$ over $U_i \subseteq P$ be local sections and $A_i = s_i^*\alpha,\ i \in \{1, 2\}$ be the corresponding local gauge potentials over U_i. Then the following holds on $U_1 \cap U_2$: $A_1 = g_{21}A_2g_{21}^{-1} + g_{21}dg_{21}^{-1}$ where $g_{21}(x) \in G$ is the uniquely defined group element with $s_2(x) = s_1(x)g_{21}(x),\ x \in U_1 \cap U_2$.

Proposition 2.7.31. Given a collection $(A_j)_{j\in J}$ of $\mathfrak{g}$-valued 1-forms with $A_j = g_{ij}A_ig_{ij}^{-1} + g_{ij}dg_{ij}^{-1},\ \forall i, j \in J$, there is a connection form α whose local gauge potentials are the A_j.

We will from now on only regard connections on principal G-bundles as being the connection forms, i.e. local Lie-algebra-valued forms.

2.7.3 Connections on Frame and Associated Bundles

We first state the general result that a connection form on a principal bundle induces a connection on its associated bundle and that a connection on a vector bundle defines a connection on its associated frame bundle.

Proposition 2.7.32. Let $\rho : G \to \mathrm{GL}(V)$ be a r-dimensional representation of the structure group G of the principal G-bundle P and let $(U_j, \varphi_j)_{j\in J}$ be a system of local trivialisations with associated local sections $s_j(a) = \varphi_j^{-1}(a, 1)$. We write $V_\rho := P \times_\rho V$ for the associated vector bundle to P. Let $\alpha \in \Omega^1(P, \mathfrak{g})$ be a connection 1-form, then the differential forms $\alpha_j := D\rho_{e_G}(s_j^*\alpha) \in \Omega^1(U_j, \mathfrak{gl}(V))$ induce a covariant derivative on V_ρ, where $D\rho_{e_G}$ is the Lie-algebra-representation of $\mathfrak{g}$ induced by ρ.

Remark 2.7.33. The differential forms α_j defined in Proposition 2.7.32 generalise the notion of local connection forms on line bundles from Proposition 2.7.8 to arbitrary dimensional vector bundles.

Remark 2.7.34. Proposition 2.7.32 gives a bijection between connections on a vector bundle E and connections on $\mathrm{Fr}(E)$.

We now restrict ourselves to line bundles and we show explicitly the bijective correspondence stated in Remark 2.7.34

Remark 2.7.35. The frame bundle associated to a line bundle $\pi\colon E \to M$ is $L^\times$, which is L with the zero section removed, i.e. we remove the zero vector of the fibres: $L^\times := L \setminus \{0_a \in L_a \mid a \in M\}$. Thus $L^\times$ is a principal $\mathbb{C}^\times$-bundle with projection $\pi : L^\times \to M$ and system of local trivialisations $(U_j, \varphi_j^\times)$ which are both restrictions of the line bundle structure of L.

Definition 2.7.36. Let $m_c\colon L^\times \to L^\times$ denote the multiplication by $c \in \mathbb{C}^\times$, i.e. $m_c(l) = lc = cl$.

Proposition 2.7.37. Let $\pi\colon L \to M$ be a line bundle with system of local trivialisations $(U_j, \varphi_j)_{j\in J}$ equipped with local connection forms $\{\alpha_j\}_{j\in J}$ as in Proposition 2.7.8. We can then use the local connection forms to define a 1-form $\alpha \in \Omega^1(L^\times, \mathbb{C})$ on the associated frame bundle $\pi\colon L^\times \to M$ defined on $L^\times|_{U_j}$ as:

$$\alpha \coloneqq \pi^*\alpha_j + \varphi_j^*\left(\frac{1}{z}dz\right),$$

where $j \in J$ and z the standard coordinate on $\mathbb{C}$. The 1-form α fulfils the following condition:

1. $m_c^*(\alpha) = \alpha,\ \forall c \in \mathbb{C}^\times$,
2. $\alpha(\eta_p) = p,\ \forall p \in \mathbb{C}$.

Thus $\alpha \in \Omega^1(M)$ is a global gauge potential (cf. Proposition 2.7.26).

Proposition 2.7.38. Conversely, every $\alpha \in \Omega^1(L^\times, \mathbb{C})$ fulfilling condition 1 and 2 from Proposition 2.7.37 induces a connection on L by

$$\nabla_X s \coloneqq s^*\alpha(X)s,\ \ s \in \Omega^0(U, L),\ \ X \in \Omega^0(U, TM).$$

2.7.4 Parallel Transport

We now define parallel transport which gives us another viewpoint on connections. It enables one to transport geometrical data from one point to another along smooth curves on the base space.

Definition 2.7.39. Let $\pi\colon E \to M$ be a vector bundle over a smooth manifold. A parallel transport system $\mathbb{P}$ on E assigns to every point $p \in E$ and every curve $\gamma\colon [a,b] \to M$ with $\gamma(a) = \pi(p)$, a unique section $\mathbb{P}_\gamma(p) \in \Omega^0_\gamma([a,b], E)$ with initial condition p, i.e. such that $\big(\mathbb{P}_\gamma(p)\big)(a) = p$. One calls $\mathbb{P}_\gamma(p)$ the parallel lift of γ starting at p. This association should satisfy the following four axioms:

1. For every smooth curve $\gamma\colon [a,b] \to M$ the map $\widehat{\mathbb{P}}_\gamma\colon E_{\gamma(a)} \to E_{\gamma(b)}$, $\widehat{\mathbb{P}}_\gamma(p) \coloneqq \big(\mathbb{P}_\gamma(p)\big)(b)$ is a linear isomorphism. Moreover $\widehat{\mathbb{P}}_\gamma^{-1} = \widehat{\mathbb{P}}_{\gamma^-}$ where $\gamma^-\colon [a,b] \to M$ is the reverse curve $t \mapsto \gamma(a - t + b)$.
2. $\mathbb{P}_\gamma(p)$ is independent of the parametrisation of the curve γ.
3. $\mathbb{P}_\gamma(p)$ depends smoothly on both γ and p.
4. Suppose $\gamma, \delta\colon [a,b] \to M$ are two curves such that $\gamma(a) = \delta(a)$ and $\gamma'(a) = \delta'(a)$. Then for each $p \in E_{\gamma(a)}$, the two curves $t \mapsto \mathbb{P}_\gamma(p)(t)$ and $t \mapsto \mathbb{P}_\delta(p)(t)$ have the same initial tangent vector:

$$\frac{d}{dt}\Big|_{t=a}\mathbb{P}_\gamma(p)(t) = \frac{d}{dt}\Big|_{t=a}\mathbb{P}_\delta(p)(t).$$

Definition 2.7.40. If $\gamma\colon [a,b] \to M$ is a smooth curve on M and let $c \in \Omega^0_\gamma([a,b],E)$ is any section along γ then we say that c is parallel along γ if $c = \mathbb{P}_\gamma(p)$ for some $p \in E_{\gamma(a)}$.

We will state that a parallel transport system induces a connection and conversely. Let us first define what horizontal lifts of curves are.

Definition 2.7.41. Let $\gamma\colon [a,b] \to M$ be a smooth curve in M. A horizontal lift of γ is a smooth curve $\lambda\colon [a,b] \to E$ such that

1. $\gamma = \pi \circ \lambda$
2. $\lambda'(t) \in \mathcal{H}_{\lambda(t)}$,

i.e. λ is horizontal along γ.

Proposition 2.7.42. Let $\pi\colon E \to M$ be a fibre bundle and let $\mathcal{H}$ be a preconnection on E. Let $\gamma\colon (a,b) \to M$ be a smooth curve and let $t_0 \in (a,b)$. Then for any $p \in E_{\gamma(t_0)}$, there exists a unique horizontal lift c of γ such that $c(t_0) = p$.

We now state the correspondence between parallel transport systems and connections.

Theorem 2.7.43. *Let $\pi\colon E \to M$ be a vector bundle and let $\mathbb{P}$ be a parallel transport system on E. Then $\mathbb{P}$ determines a connection $\mathcal{H}$ on E by the following property: a section c along a curve γ is parallel in the sense of Definition 2.7.40 if and only if c is horizontal with respect to $\mathcal{H}$ in the sense of Definition 2.7.16.*

Remark 2.7.44. The intuition behind the correspondence between parallel sections and horizontal sections of Theorem 2.7.43 relies on the uniqueness of horizontal sections from Proposition 2.7.42 and on the uniqueness of parallel sections from Definition 2.7.39.

Conversely a connection induces a parallel transport.

Theorem 2.7.45. *Let $\pi\colon E \to M$ be a vector bundle and let $\mathcal{H}$ be a connection on E. The system of all horizontal lifts to E of smooth curves in E defines a parallel transport system $\mathbb{P}$ in E.*

Remark 2.7.46. One can prove the following direct relation between a covariant derivative ∇ on E and the corresponding parallel transport system

$$(\nabla_X s)(x) = \lim_{h\to 0} \frac{\widehat{\mathbb{P}}_\gamma(s \circ \gamma(h)) - s \circ \gamma(0)}{h}, \;\; \gamma(0) = x, \gamma'(0) = X(x), s \in \Omega^0(M,E).$$

Therefore the covariant derivative ∇_X measures to what extent the section s deviates from being horizontal along the curve γ. To see this, recall from Remark 2.7.20 that a section is horizontal if and only if $\nabla_X s = 0$ and $\widehat{\mathbb{P}}_\gamma(s\circ\gamma(h)) - s\circ\gamma(0)$ vanishes when $s \circ \gamma$ is parallel along γ.

2.8 Curvature

Curvature will give us one of the prequantisation criteria.

Definition 2.8.1. Let $\pi : E \to M$ be a vector bundle with connection ∇. We say that ∇ is a flat connection if the corresponding distribution $\mathcal{H}$ of E is integrable. The pair (E, ∇) is referred to as a flat vector bundle.

Definition 2.8.2. Let $\pi : E \to M$ be a vector bundle with connection ∇. The curvature tensor R^∇ of ∇ is defined as follows. For vector fields $X, Y \in \Omega^0(M, TM)$ and $s \in \Omega^0(M, E)$ define R^∇ by setting

$$R^\nabla(X, Y)(s) := [\nabla_X, \nabla_Y](s) - \nabla_{[X,Y]}(s).$$

Remark 2.8.3. The curvature form Ω is defined as $\Omega(X, Y) := R^\nabla(X, Y),\ X, Y \in \Omega^0(M, TM)$. One can show that this curvature form is an element of $\Omega^2(M, E)$ and that on a vector bundle with local trivialisation system $(U_j, \varphi_j)_{j\in J}$ it is induced by the local connection forms $\{\alpha_j\}_{j\in J}$ as follows: $\Omega|_{U_j} := d\alpha_j + \alpha_j \wedge \alpha_j,\ j \in J$. In particular, $\alpha_j \wedge \alpha_j$ vanishes on a line bundle and the curvature form becomes $\Omega|_{U_j} = d\alpha_j$. We will also denote the curvature form as $\mathrm{Curv}(E, \nabla)$ since it is specific to a certain vector bundle and it is related to the connection. By abuse of notation, we will write $\mathrm{Curv}(\nabla)$, since a certain connection lives on a specific vector bundle.

Proposition 2.8.4. The following properties are equivalent:

1. ∇ is a flat connection.
2. The corresponding parallel transport is locally independent of the curves.
3. The curvature equals zero, i.e. $R^\nabla(X, Y) = 0,\ X, Y \in \Omega^0(U, TM)$.

We now define what a curvature form on a principal bundle is.

Definition 2.8.5. Let $\pi : P \to M$ be a principal G-bundle with G-connection given by connection form $\alpha \in \Omega^1(P, \mathfrak{g})$. The curvature form $\Omega \in \Omega^2(P, \mathfrak{g})$ is given by

$$\Omega := d\alpha + \frac{1}{2}[\alpha, \alpha], \tag{2.4}$$

where $[\cdot, \cdot]$ denotes the product of Lie algebra-valued forms from Example 2.5.38.

We now restrict ourselves to line bundles to show how curvature forms on principal $\mathbb{C}^\times$-bundles $L^\times$ relate to curvature forms on their associated vector bundles L and conversely.

Remark 2.8.6. Note that on $L^\times$ the curvature form takes the form $d\alpha$. Indeed the connection form α belongs to $\Omega^1(L^\times, \mathbb{C})$ and therefore $[\alpha, \alpha]$ in equation (2.4) vanishes (see Example 2.5.38 and recall that the only possible Lie bracket on $\mathbb{C}$ is the trivial bracket). We will thus denote $d\alpha$ the connection form on $L^\times$ and Ω the connection form on L.

Proposition 2.8.7. Let $\widetilde{\pi}\colon L \to M$ be a line bundle and $\pi : L^\times \to M$ the associated principal $\mathbb{C}^\times$-bundle, then the following hold:

1. $\pi^*\Omega = d\alpha$ and
2. conversely, $s^* d\alpha = \Omega|_U$ for any section $s \in \Omega^0(U, L^\times)$.

2.9 Integration

We will now discuss integration on orientable smooth manifolds. As mentioned earlier, differential forms are going to be very important to make sense of integration.

We will first discuss orientability of smooth manifolds since we can only integrate on orientable smooth manifolds.

Let us first define what orientable means for vector spaces.

Definition 2.9.1. Let V be a one-dimensional real vector space. Then $V \setminus \{0\}$ has two components. An orientation of V is a choice of one of these components, which one labels positive. The other one will be denoted negative.

Definition 2.9.2. The determinant of V is $\Lambda^n(V)$ where $n = \dim(V)$. We will denote it with $\det V$. An orientation on V is then a choice of orientation on $\det V$.

We now define what orientable means for vector bundles.

Definition 2.9.3. Let $\pi\colon E \to M$ be a vector bundle. The determinant line bundle associated to E is the vector bundle $\det E \to M$ of rank one whose fibre over $x \in M$ is $\det E_x$.

Remark 2.9.4. The above definition of the determinant line bundle is sensible by the Metathorem, since it doesn't involve a choice of basis.

Definition 2.9.5. A vector bundle E is orientable if there is a smooth nowhere vanishing section $\mu \in \Omega^0(M, \det E^*)$.

Remark 2.9.6. We note that from Proposition 2.4.34 the condition from Definition 2.9.5 is equivalent to the line bundle $\det E^* \to M$ being trivialisable.

Definition 2.9.7. Let μ be a non-vanishing section of $\det E^*$. Then the equivalence class $[\mu]$ is called an orientation, where the equivalence relation is defined as follows. Let μ and η be two non-vanishing sections of $\det E^*$. The two sections are equivalent if there exists a smooth function $f\colon M \to (0, \infty)$ such that $\mu = f\eta$. Additionally if we set $[\mu]$ to be the positive orientation, then $h\mu$ with $h\colon M \to (-\infty, 0)$ defines the negative orientation.

Definition 2.9.8. A smooth manifold M is said to be orientable if the tangent bundle TM is an orientable vector bundle.

Definition 2.9.9. A volume form on an n-dimensional smooth manifold M is a nowhere vanishing differential n-form.

Remark 2.9.10. Since $\Omega^0(M, \det T^*M)$ equals $\Omega^n(M)$, we understand that μ from Definition 2.9.7 is a nowhere vanishing form of top degree. These volume forms will give us the mass which we will integrate. A smooth manifold M with a choice of orientation $[\mu]$ will be denoted $(M, [\mu])$ and we will say that it is an oriented manifold.

Remark 2.9.11. Let $(M, [\mu])$ and $(N, [\nu])$ be two oriented manifolds of the same dimension n. Suppose $\phi\colon M \to N$ is a diffeomorphism. Then there exists a smooth nowhere vanishing function $f \in C^\infty(M)$ such that $\phi^*(\nu) = f\mu$. Indeed we have $(\phi^*(\nu))_p(v) = \nu_{\phi(p)}(D\phi_p v)$. We know that ν is nowhere vanishing and we also know that $D\phi_p$ is an isomorphism. Thus for $v \neq 0$, $\phi^*(\nu)$ is nowhere vanishing. The existence of the function is due to the fact that $\Omega^n(M)$ is the space of sections of a one-dimensional bundle (see Example 2.4.33). We say that ϕ is orientation preserving if f is everywhere positive, i.e. $[\phi^*(\nu)] = [\mu]$.

Definition 2.9.12. A chart $\sigma\colon U \to O$ on an oriented smooth manifold M of dimension n is said to be positively oriented if σ is an orientation preserving diffeomorphism between manifolds U and O, where U inherits the orientation of M and O inherits the standard orientation from $\mu \coloneqq dx_1 \wedge \cdots \wedge dx_n$, where $x_1, ..., x_n$ are the standard coordinates of $\mathbb{R}^n$.

We now define integration, first for open subsets U of $\mathbb{R}^n$ and then more generally for oriented manifolds.

Definition 2.9.13. The support of a differential form $\alpha \in \Omega^n(M)$ is defined as the set $\{p \in M \mid \alpha(p) = 0\}$. We denote it by $\mathrm{supp}(\alpha)$.

Definition 2.9.14. Let $U \subseteq \mathbb{R}^n$ be open with standard coordinates $(x_1, ..., x_n)$ and $\alpha \in \Omega^n(\mathbb{R}^n)$. From Proposition 2.5.27 we may write $\alpha = f dx_1 \wedge \cdots \wedge dx_n$ for some function f. Suppose $\mathrm{supp}(\alpha) \cap U$ is compact. We define the integral of α on U by

$$\int_U \alpha = \int_U f dx_1 \wedge \cdots \wedge dx_n \coloneqq \int_U f dx_1 \cdots dx_n$$

where we use for the last term the Lebesque integral, with respect to the Lebesque measure $dx_1 ... dx_n$ on $\mathbb{R}^n$.

Definition 2.9.15. Let M be an oriented manifold of dimension n and let $\alpha \in \Omega^n(M)$. Assume that α has compact support inside U for some positively oriented chart (U, σ). One can show that $(\sigma^{-1})^*\alpha$ have compact support on $\sigma(U)$. We define the integration of α on U as follows

$$\int_U \alpha \coloneqq \int_{\sigma(U)} (\sigma^{-1})^* \alpha.$$

Example 2.9.16. We will be mostly interested in integrating on symplectic manifolds, which will describe the classical system we will consider. A symplectic manifold M of dimension $2n$ endowed with a symplectic form ω is orientable. Indeed we can define the following differential form of top degree:

$$\mu \coloneqq \frac{1}{n!} \underbrace{\omega \wedge \cdots \wedge \omega}_{n-\text{times}} = \frac{\omega^{\wedge n}}{n!}$$

which is nowhere vanishing since ω is non-degenerate. This differential form μ is called the Liouville volume form.

For the symplectic manifold (M, ω) from Example 2.6.5, the Liouville volume form is $\mu = \omega$, because $n = 1$.

We now state Stokes' theorem. We will not discuss manifolds with boundaries. More about this topic can be found in [1] and in [3]. We just recall that the boundary ∂M naturally comes with the structure of a smooth orientable manifold of dimension $n - 1$. Additionally, if M is orientable then ∂M receives an induced orientation.

Theorem 2.9.17. *(Stokes' Theorem) Let M be an oriented smooth manifold of dimension n with boundary and give ∂M the induced orientation. Then for any $\omega \in \Omega^{n-1}(M)$ with compact support we have*

$$\int_M d\omega = \int_{\partial M} \iota^* \omega,$$

where $\iota \colon \partial M \hookrightarrow M$.

Remark 2.9.18. In particular, $\int_M d\omega = 0$ if $\partial M = \varnothing$.

Chapter 3

Hamiltonian Mechanics

We will first describe Hamiltonian mechanics on generic symplectic manifolds. We will then specialise to vector spaces.

3.0.1 Hamiltonian system

For the following, (M, ω) denotes a symplectic manifold of dimension $2n$, which we call phase space in the context of Hamiltonian mechanics.

Proposition 3.0.1. The musical maps $\omega^\flat$ and its inverse $\omega^\sharp$ are vector bundle isomorphisms. They are defined as follows:

$$\omega^\flat \colon TM \to T^*M, \ \omega^\flat(X)(Y) := \omega(X, Y), \ X, Y \in TM$$

and its inverse is defined as:

$$\omega^\sharp \colon T^*M \to TM, \ \omega^\sharp := (\omega^\flat)^{-1}.$$

Proof (Sketch). We have to show that $\omega^\flat$ is an isomorphism and that it is smoothly varying on M. Since $\omega^\flat$ is linear and the dimensions of TM and T^*M are the same, it is sufficient to show that $\omega^\flat_p$ is injective for all $p \in M$. The kernel of $\omega^\flat_p$ is given by:

$$\mathrm{Ker}(\omega^\flat_p) = \{X \in T_pM \mid \omega(X, Y) = 0, \ \forall Y \in T_pM\}.$$

From the non-degeneracy of the symplectic form ω it follows that $\mathrm{Ker}(\omega^\flat_p) = \{0\}$. The proof that $\omega^\flat$ is smoothly varying on M is left to the reader. □

Remark 3.0.2. In local coordinates, the musical isomorphisms are given as follows. Let $(x_1, ..., x_{2n})$ denote local coordinates on $U \subseteq M$. We can write the symplectic form on U locally as $\omega|_U = \sum_{ij} \omega_{ij} dx_i \wedge dx_j, \ \omega_{ij} \in C^\infty(U)$.

For a vector field $X \in \Omega^0(U, TM)$ written $X = \sum_k X_k \frac{\partial}{\partial x_k}$, the flat musical isomorphism on U applied on X yields:

$$\omega^\flat(X) = \omega(X, \cdot) = \sum_{ij} \omega_{ij} X_i dx_j =: \sum_j \omega^\flat(X)_j dx_j.$$

For a 1-form $\alpha \in \Omega^1(U)$, the sharp musical isomorphism applied on $\alpha = \sum_k \alpha_k dx_k$ yields:

$$\omega^\sharp(\alpha) = \sum_{ji} \omega_{ij}^{-1} \alpha_i \frac{\partial}{\partial x_j} =: \sum_j \omega^\sharp(\alpha)_j \frac{\partial}{\partial x_j}. \tag{3.1}$$

With this we can associate to each function its Hamiltonian vector field. This vector field will enable us to describe the dynamics of a classical system.

Definition 3.0.3. To every function $H \in C^\infty(M)$, the Hamiltonian vector field X_H of H is defined as:

$$X_H := \omega^\sharp \circ dH. \tag{3.2}$$

The triple (X, ω, H) is called a Hamiltonian system.

Remark 3.0.4. Equivalently, we find $dH = \omega^\flat X_H$ by applying $\omega^\flat$ on the left of equation (3.2). Thus X_H is the uniquely defined vector field satisfying

$$\omega(X_H, Y) = \omega^\flat(X_H)(Y) = dH(Y) = Y.H,\ Y \in \Omega^0(M, TM) \tag{3.3}$$

In local coordinates $(x_1, ..., x_{2n})$ on $U \subseteq M$, the Hamiltonian vector field takes the form:

$$X_H|_U = \sum_{ji} \omega_{ij}^{-1} \frac{\partial H}{\partial x_i} \frac{\partial}{\partial x_j}, \tag{3.4}$$

by using equation 3.1 and $dH = \sum_k \frac{\partial H}{\partial x_k} dx_k$.

Definition 3.0.5. A trajectory of the Hamiltonian system (M, ω, H) is a curve $\gamma(t) = (x_1(t), ..., x_n(t))^T \in M$ such that $\gamma'(t) = X_H(\gamma(t)),\ \forall t \in I \subseteq \mathbb{R}$. This system of 1st order non-linear ordinary differential equations are called the equations of motion.

Remark 3.0.6. Let $(q_1, .., q_n, p_1, ..., p_n)$ on $U \subseteq M$ be local Darboux coordinates. Recall that in Darboux coordinates the symplectic form is given by $\omega|_U = \sum_j dq_j \wedge dp_j$. Using equation (3.4), the equations of motion take the form:

$$q'(t) = \frac{\partial H}{\partial p},\ p'(t) = -\frac{\partial H}{\partial q},$$

if $q(t) := q(\gamma(t)), p(t) := p(\gamma(t))$, where $(q, p)\colon U \to O \subseteq \mathbb{R}^n \times \mathbb{R}^n$ is the canonical chart. These equations are called the canonical Hamiltonian equations.

3.0.2 Poisson brackets

Hamiltonian mechanics is usually described with the use of Poisson brackets. Poisson brackets are actually Lie brackets as we will see below.

Definition 3.0.7. The Poisson bracket associated to a symplectic form ω is defined as follows

$$\{f, g\} \coloneqq \omega(X_f, X_g),\ f, g \in C^\infty(M).$$

Remark 3.0.8. From Remark 3.0.4, we can write with the help of Poisson brackets:

$$\{f, g\} = X_g.f,\ g, f \in C^\infty(M). \tag{3.5}$$

In local Darboux coordinates, the Poisson bracket takes the form:

$$\{f, g\} = \sum_j \frac{\partial g}{\partial p_j}\frac{\partial f}{\partial q_j} - \frac{\partial g}{\partial q_j}\frac{\partial f}{\partial p_j}.$$

Proposition 3.0.9. The Poisson bracket $\{\cdot,\cdot\}$ of a symplectic manifold (M, ω) is a Lie bracket i.e. $C^\infty(M)$ with $\{\cdot,\cdot\}$ is a Lie algebra over $\mathbb{R}$. In addition $\{f,\cdot\}\colon C^\infty(M) \to C^\infty(M)$ is a derivation on the $\mathbb{R}$-algebra $C^\infty(M)$, i.e. it fulfils the product rule

$$\{f, gh\} = g\{f, h\} + \{f, g\}h = g\{f, h\} + h\{f, g\},\ f, g, h \in C^\infty(M).$$

Definition 3.0.10. A classical observable is an element of the $\mathbb{R}$-algebra $(C^\infty(M), \{\cdot,\cdot\})$.

Remark 3.0.11. The most important classical observable on a Hamiltonian system is the Hamiltonian H which measures the energy of the system. Position and momentum are two other examples of classical observables.

Proposition 3.0.12. A trajectory $\gamma\colon (t_0, t_1) \to M$ is a solution of $\gamma'(t) = X_H(\gamma(t))),\ \forall t$ if and only if

$$f' = \{f, H\},\ \forall f \in C^\infty(M),$$

i.e.

$$\frac{d}{dt} f(\gamma(t)) = \{f(\gamma(t)), H(\gamma(t))\},\ \forall t \in (t_0, t_1).$$

Remark 3.0.13. The equation from Proposition 3.0.12 is going to be central in the prequantisation procedure. Indeed this equation fixes the dynamics of the observables on the Hamiltonian system. During the quantisation process, we would like to construct quantum operators which inherit the dynamics of the classical observables. The following proposition will be useful to construct these quantum operators. It relates the Lie bracket of the vector fields Lie algebra to the Poisson bracket of the $C^\infty(M)$ Lie algebra.

Proposition 3.0.14. The map $\Phi\colon C^\infty(M) \to \Omega^0(M, TM),\ f \mapsto -X_f$ is a Lie algebra homomorphism, i.e.

$$[X_f, X_g] = -X_{\{f,g\}},\ \forall f, g \in C^\infty(M).$$

Proof. Let $f, g, h \in C^\infty(M)$ be smooth functions on M. We then compute:

$$\begin{aligned}
[X_f, X_g].h &= X_f.(X_g.h) - X_g.(X_f.h) \\
&= \{X_g.h, f\} - \{X_f.h, g\} \\
&= \{\{h, g\}, f\} - \{\{h, f\}, g\} \\
&= \{f, \{g, h\}\} + \{g, \{h, f\}\} \\
&\overset{(1)}{=} -\{h, \{f, g\}\} \\
&= -X_{\{f,g\}}.h,
\end{aligned}$$

where we used the Jacobi identity in (1) and equation (3.5) several times. □

3.0.3 The 2-dimensional Flat Hamiltonian System

Let us discuss a Hamiltonian system on the 2-dimensional symplectic manifold from Example 2.6.5. We recall that we had global canonical coordinates q and p on $M = T^*\mathbb{R}$ and the symplectic form was defined as $\omega = dq \wedge dp$. From equation (3.4) and the matrix representation of the symplectic form, we see that the Hamiltonian vector field is globaly given by:

$$\begin{aligned}
X_H &= (\omega_{12})^{-1} \frac{\partial H}{\partial q} \frac{\partial}{\partial p} + (\omega_{21})^{-1} \frac{\partial H}{\partial p} \frac{\partial}{\partial q} \\
&= -\frac{\partial H}{\partial q} \frac{\partial}{\partial p} + \frac{\partial H}{\partial p} \frac{\partial}{\partial q},
\end{aligned} \tag{3.6}$$

since $(\omega_{11})^{-1} = (\omega_{22})^{-1} = 0$. Thus the equations of motion take the form:

$$(q', p') = \left(\frac{\partial H}{\partial q}, -\frac{\partial H}{\partial p} \right),$$

where $q(t) \coloneqq q(\gamma(t)),\ p(t) \coloneqq p(\gamma(t))$ and $\gamma(t)$ a trajectory of the system. These equations of motion obviously agree with the local equations of motion from Remark 3.0.6.
Finally we compute the Hamiltonian vector fields to the classical observables of position and momentum by inputting q, p instead of H in equation (3.6):

$$X_q = -\frac{\partial}{\partial p},\ X_p = \frac{\partial}{\partial q}.$$

Chapter 4

Quantum Mechanics

We will present the basics of quantum mechanics in the optic of geometric quantisation. We discuss the axioms of quantum mechanics, the Schrödinger position and momentum representation, the Heisenberg uncertainty principle, the Stone–von Neumann theorem and the Heisenberg picture of quantum mechanics.

4.1 The Axioms of Quantum Mechanics

We describe here the fundamental assumptions of quantum mechanics which were motivated by experiments. We will denote these assumptions as axioms and only cite the ones which are relevant for quantisation. One can find a further discussion in [4].
Quantum mechanics is a microscopic description of nature. The fundamental idea of quantum mechanics is that a physical system can only be described probabilistically. This probabilistic description of a physical system is encoded in a pure state (we will not discuss mixed states), i.e. a line in a Hilbert space, the so called quantum Hilbert space. Let us now formally describe this idea with the following axioms.

Axiom 1. The state of the system is represented by a unit vector ψ (up to a phase) in a Hilbert space $\mathcal{H}$ with inner product $< \cdot \mid \cdot >$. We call ψ the wave function of the system.

Remark 4.1.1. We will use the convention that the inner product of the quantum Hilbert space is linear in the first component and antilinear in the second.

Example 4.1.2. For a particle living in $\mathbb{R}^3$, the quantum Hilbert space is $L^2(\mathbb{R}^3)$ with inner product

$$< \psi \mid \phi > := \int_{\mathbb{R}^3} \psi(\vec{x})\overline{\phi(\vec{x})} d^3\vec{x}, \ \psi, \phi \in L^2(\mathbb{R}^3)$$

and the wave functions ψ are unit vectors in $L^2(\mathbb{R}^3)$. This is sensible as the probability distribution of the position of the particle is actually given by $|\psi(\vec{x})|^2$, as we will understand after axiom 3, and therefore the following should hold

$$1 = \int_{\mathbb{R}^3} |\psi|^2 d^3\vec{x} = \int_{\mathbb{R}^3} \psi(\vec{x})\overline{\psi(\vec{x})} d^3\vec{x} =< \psi \mid \psi > .$$

Before stating the second axiom, we define what is a self-adjoint operator on the Hilbert space $\mathcal{H}$. An operator on $\mathcal{H}$ is a linear map $A : \mathrm{Dom}(A) \to \mathcal{H}$, where $\mathrm{Dom}(A) \subseteq \mathcal{H}$ denotes the dense subspace of $\mathcal{H}$ where A is defined. An operator is bounded if there exists $C \in \mathbb{R}$ such that $\|A\psi\| \leq C\|\psi\|$, for any unitary vector $\psi \in \mathcal{H}$. If an operator is bounded we can extend it by continuity to the whole of $\mathcal{H}$. However, the operators that we will encounter will, in general, be unbounded. We then define the adjoint of an unbounded operator.

Definition 4.1.3. Let A be an operator on $\mathcal{H}$, the adjoint operator A^* of A is defined as follows. A vector $\varphi \in \mathcal{H}$ belongs to the domain $\mathrm{Dom}(A^*)$ of A^* if the linear functional $< \varphi \mid A\cdot >$, defined on $\mathrm{Dom}(A)$, is bounded. For $\varphi \in \mathrm{Dom}(A^*)$, $A^*\varphi$ is the unique vector χ such that $< \chi \mid \psi >=< \varphi \mid A\psi >$ for all $\psi \in \mathrm{Dom}(A)$.

We can now define what self-adjoint means.

Definition 4.1.4. An operator A on $\mathcal{H}$ is symmetric if

$$< \varphi \mid A\tilde{\varphi} >=< A\varphi \mid \tilde{\varphi} >$$

for all $\varphi, \tilde{\varphi} \in \mathrm{Dom}(A)$. The operator A is self-adjoint if $\mathrm{Dom}(A^*) = \mathrm{Dom}(A)$ and $A^*\varphi = A\varphi$ for all $\varphi \in \mathrm{Dom}(A)$.

Remark 4.1.5. As stated above, if an operator A is bounded, we can then automatically extend $\mathrm{Dom}(A)$ to $\mathcal{H}$. We can extend $\mathrm{Dom}(A^*)$ to $\mathcal{H}$ as well. Thus, A is symmetric if and only if A is self-adjoint.

Axiom 2. To each classical observable f on the classical phase space there is an associated self-adjoint operator $\widehat{f}$ on the quantum Hilbert space $\mathcal{H}$. We call such an operator a quantum observable.

Remark 4.1.6. A quantum system is a quantum Hilbert space and a preferred choice of quantum observable H, the Hamiltonian.

The third axiom explains how the theory of quantum mechanics is linked to experiments.

Axiom 3. If a quantum system is in a state described by a unit vector $\psi \in \mathcal{H}$, the expected value of the quantum mechanical observable $\widehat{f}$ in the state ψ is given by $\mathbb{E}_\psi(\widehat{f}) =< \psi \mid \widehat{f}\psi >$.

Let us now give a motivation for Axiom 2. Per definition, a self-adjoint operator is in particular a symmetric operator. For symmetric operators the following proposition holds.

Proposition 4.1.7. Suppose A is a symmetric operator on $\mathcal{H}$.

- For all $\psi \in \mathrm{Dom}(A)$, the quantity $< \psi \mid A\psi >$ is real.
- Suppose λ is an eigenvector for A, meaning that $A\psi = \lambda\psi$ for some nonzero $\psi \in \mathrm{Dom}(A)$. Then $\lambda \in \mathbb{R}$.

Proof. Since A is symmetric, the following calculataion holds for $\psi \in \mathrm{Dom}(A)$, $< \psi \mid A\psi >=< A\psi \mid \psi >= \overline{< \psi \mid A\psi >}$ and hence, $< \psi \mid A\psi >$ is real. If ψ is an eigenvector of A with eigenvalue λ, then $\lambda < \psi \mid \psi >=< A\psi \mid \psi >=< \psi \mid A\psi >= \overline{\lambda} < \psi \mid \psi >$, whence $\lambda = \overline{\lambda}$. □

In physics, the expectation value of a measurement should be real as we only measure real values. Therefore, Proposition 4.1.7 guarantees that we will have operators which are physically sensible. The stronger constraint of self-adjointness is required because of the spectral theorem which is discussed in [4].

4.2 The Position and Momentum Operators

Now that we understand the foundations of quantum mechanics, we shall discuss two of the most important operators, the momentum and position operators.

Definition 4.2.1. For a particle moving in $\mathbb{R}^3$, let $L^2(\mathbb{R}^3)$ be the quantum Hilbert space of Example 4.1.2. The position operators $(\widehat{q}_i)_{i\in\{1,2,3\}}$ and momentum operator $(\widehat{p}_i)_{i\in\{1,2,3\}}$ are given by

$$\widehat{q}_i\psi(\vec{x}) = x_i\psi(\vec{x}),$$
$$\widehat{p}_i\psi(\vec{x}) = -i\hbar\frac{\partial}{\partial x_i}\psi(\vec{x}),$$

where $\psi(\vec{x}) \in L^2(\mathbb{R}^3)$, $\hbar$ is the reduced Planck constant.

Remark 4.2.2. The Planck constant $h = 2\pi\hbar$ is the quantum of action. It is an experimentally measured constant, which relates for example the momentum p of a photon to the frequency k of its wave function via $p = \hbar k$, according to the de Broglie hypothesis (see [4]).

Remark 4.2.3. One can easily show that the position and momentum operators from Definition 4.2.1 are symmetric (clear for the position operator and integration by part for the momentum operator). It is however more complicated to show that the operators are (essentialy) self-adjoint, see [4] Chapter 9.

The following relation between the two operators is crucial as it enforces the Heisenberg uncertainty principle that we will discuss below.

Proposition 4.2.4. The position and momentum operators $(\widehat{q}_i)_{i\in\{1,2,3\}}$ and $(\widehat{p}_i)_{i\in\{1,2,3\}}$ satisfy the commutation relation

$$[\widehat{q}_k, \widehat{p}_l] = \widehat{q}_k\widehat{p}_l - \widehat{p}_l\widehat{q}_k = i\hbar\delta_{kl}.$$

We call it the canonical commutation relation.

Proof. With the product rule, we can calculate the following

$$\begin{aligned}
[\widehat{q}_k, \widehat{p}_l]\psi &= -i\hbar x_k \frac{\partial}{\partial x_l}\psi(\vec{x}) + i\hbar \frac{\partial}{\partial x_l} x_k \psi(\vec{x}) \\
&= -i\hbar x_k \frac{\partial}{\partial x_l}\psi(\vec{x}) + i\hbar \delta_{kl}\psi(\vec{x}) + i\hbar x_k \frac{\partial}{\partial x_l}\psi(\vec{x}) \\
&= i\hbar \delta_{kl}\psi(\vec{x}).
\end{aligned}$$

□

We now define the uncertainty of a quantum observable and the Heisenberg uncertainty principle.

Definition 4.2.5. If A is a quantum operator on a Hilbert space $\mathcal{H}$ and ψ is a unit vector in $\mathcal{H}$, the standard deviation $\Delta_\psi A \in \mathbb{R}$ associated with the measurement of A in the state ψ is given by

$$(\Delta_\psi A)^2 = \mathbb{E}_\psi(A^2) - \mathbb{E}_\psi(A)^2.$$

We call $\Delta_\psi A$ the uncertainty of A in the state ψ.

Proposition 4.2.6. (Heisenberg uncertainty principle) The uncertainties of the position and momentum operators are related to each other in the following way:

$$(\Delta_\psi \widehat{q}_i)(\Delta_\psi \widehat{p}_j) \geq \left| \frac{1}{2i}\mathbb{E}_\psi([\widehat{q}_i, \widehat{p}_j]) \right| = \frac{\hbar}{2}\delta_{ij}.$$

A more precise uncertainty principle is given by the Robertson principle. The derivation of this principle can be found in [4]. The interpretation of the Heisenberg uncertainty principle is that it is impossible to measure simultaneously position and momentum with infinite precision. This will be central when constructing a quantum Hilbert space in the quantisation process, see Remark 5.3.7.

4.3 The Stone–von Neumann Theorem

In this section we describe the Schrödinger representation of quantum mechanics and its uniqueness given by the Stone–von Neumann theorem.
Let us first discuss the Heisenberg group.

Definition 4.3.1. The continuous Heisenberg group is the subgroup of $\mathrm{Gl}_3(\mathbb{R})$ given by the matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix},$$

where $a, b, c \in \mathbb{R}$.

Remark 4.3.2. One can show that the continuous Heisenberg group is a Lie group and its Lie algebra $\mathfrak{h}$ is generated by the matrices of the form

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The generators of the Lie algebra fulfil the commutation relations $[X, Y] = Z$, $[X, Z] = 0$ and $[Y, Z] = 0$.

We will now define what a universal enveloping algebra is.

Definition 4.3.3. Let $\mathfrak{g}$ be a Lie algebra. The universal enveloping algebra of $\mathfrak{g}$, denoted $U(\mathfrak{g})$, is an associative algebra with a Lie morphism $\iota\colon \mathfrak{g} \hookrightarrow U(\mathfrak{g})$ satisfying the following universal property. If $\varphi\colon \mathfrak{g} \to A$ is a morphism, where A is an associative algebra, there is a unique morphism of associative algebras $U(\varphi)\colon U(\mathfrak{g}) \to A$ such that $\iota = \varphi \circ U(\varphi)^{-1}$.

Remark 4.3.4. The universal enveloping algebra is unique up to isomorphism because of the universal property. Indeed, suppose that B and B' are associative algebras with Lie morphisms $\iota_B\colon \mathfrak{g} \hookrightarrow B$ and $\iota_{B'}\colon \mathfrak{g} \hookrightarrow B'$ satisfying the universal property. We can then consider B' as the equivalent to A in Definition 4.3.3 which yields a unique morphism of associative algebras $U(\iota_{B'})$ satisfying $\iota_B = \iota_{B'} \circ U(\iota_{B'})^{-1}$. Similarly, we find another unique morphism of associative algebras $U(\iota_B)$ satisfying $\iota_{B'} = \iota_B \circ U(\iota_B)^{-1}$. The two morphisms $U(\iota_{B'})$ and $U(\iota_B)$ satisfy $U(\iota_B) \circ U(\iota_{B'}) = I_{B'}$ as well as $U(\iota_{B'}) \circ U(\iota_B) = I_B$.

Remark 4.3.5. We can construct the universal enveloping algebra for an arbitrary Lie algebra $\mathfrak{g}$ via the following procedure. Consider the tensor algebra $\mathrm{Tens}(\mathfrak{g}) \coloneqq \bigoplus_{k \geq 0} g^{\otimes k}$. This algebra is a non commutative associative algebra over $\mathbb{C}$ with the product defined on monomials by:

$$\begin{gathered} \cdot\colon \mathrm{Tens}(\mathfrak{g}) \times \mathrm{Tens}(\mathfrak{g}) \to \mathrm{Tens}(\mathfrak{g}) \\ x_1 \otimes \cdots \otimes x_n \cdot y_1 \otimes \cdots y_m \mapsto x_1 \otimes \cdots \otimes x_n \otimes y_1 \otimes \cdots y_m, \end{gathered}$$

where $n, m \in \mathbb{N}$. This definition can be extended by $\mathbb{C}$-linearity to any element of $\mathrm{Tens}(\mathfrak{g})$. We can then consider the 2-sided ideal $I \subseteq \mathrm{Tens}(\mathfrak{g})$ generated by:

$$x \otimes y - y \otimes x - [x, y],$$

for $x, y \in \mathfrak{g}$. One can show that the universal enveloping algebra of $\mathfrak{g}$ can be defined as $U(\mathfrak{g}) \coloneqq \mathrm{Tens}(\mathfrak{g})/I$.

Let us now construct the universal enveloping algebra $U(\mathfrak{h})$ of the Heisenberg Lie algebra $\mathfrak{h}$:

$$U(\mathfrak{h}) \coloneqq \mathrm{Tens}(\mathfrak{h})/(XY - YX - Z, XZ - ZX, YZ - ZY).$$

This implies that X and Y as well as Y and Z commute in $U(\mathfrak{h})$. It also implies that $XY - YX = Z$ in $U(\mathfrak{h})$.

Remark 4.3.6. We remark that Z is central in $U(\mathfrak{h})$, i.e. $Z \in \{a \in U(\mathfrak{h}) \mid ab - ba = 0,\ \forall b \in U(\mathfrak{h})\}$.

For $\hbar \in \mathbb{C}$, consider the bilateral ideal $I_\hbar$ generated by $z - i\hbar$. Then $U(\mathfrak{h})/I_\hbar$ is isomorphic to the 1-dimensional Weyl algebra W, see [10] Example 1.10. The 1-dimensional Weyl algebra is particularly important in quantum mechanics as the position and momentum operators on $L^2(\mathbb{R})$, discussed in the previous sections, are a representation of the Weyl algebra. Indeed, the representation ρ_{SP} of $L^2(\mathbb{R})$ is the morphism $\rho_{SP}\colon W \to \mathrm{End}(L^2(\mathbb{R}))$ given by:

$$\rho_{SP}(X) = \widehat{q} = x,\ \rho_{SP}(Y) = \widehat{p} = -i\hbar\frac{\partial}{\partial x}$$

We call $(\rho_{SP}, L^2(\mathbb{R}))$ the Schrödinger position representation. In 3-dimensions, the Heisenberg Lie algebra is spanned by elements $(X_i, Y_i)_{i=1,\cdots,3}$ and Z such that they fulfil the relations $[X_i, Y_j] = Z\delta_{ij}$, $[X_i, Z] = 0$ and $[Y_i, Z] = 0$, $i, j \in \{1, 2, 3\}$. The Schrödinger position representation on $L^2(\mathbb{R}^3)$ is then the morphism $\rho_{SP}\colon W \to \mathrm{End}(L^2(\mathbb{R}^3))$ given by:

$$\rho_{SP}(X_j) = \widehat{q}_j = x_j,\ \rho_{SP}(Y_j) = \widehat{p}_j = -i\hbar\frac{\partial}{\partial x_j},\ j \in \{1, 2, 3\}.$$

We can now state the Stone–von Neumann theorem without getting into the technical details related to the unboundedness of the operators, since these will not be important for quantisation. One can find a full discussion of the theorem and its proof in [4], Chapter 14.

Theorem 4.3.7. *(Stone–von Neumann) There exists a unique irreducible representation of the Weyl algebra W on a Hilbert space up to unitary transformation.*

Therefore ρ_{SP} is unique up to unitary transformation. The following example describes another representation which is unitary equivalent to the Schrödinger position representation.

Example 4.3.8. The Schrödinger momentum representation $\rho_{SM} : W \to \mathrm{End}_{\mathbb{C}}(L^2(\mathbb{R}^3))$ is given by

$$\rho_{SM}(X_i) = \widehat{q}_i = i\hbar\frac{\partial}{\partial \xi_i},\ \rho_{SM}(Y_i) = \widehat{p}_i = \xi_i,$$

where we have chosen $\vec{\xi} = (\xi_1, \xi_2, \xi_3)$ as the coordinate of $\mathbb{R}^3$. This representation is equivalent to the Schrödinger position representation and the unitary transformation that intertwines these operators is the Fourier transformation $\mathcal{F}$, i.e. $\rho_{SM} \circ \mathcal{F} = \mathcal{F} \circ \rho_{SP}$.

Another example of a representation which is unitary equivalent to the Schrödinger position representation is the Fock representation on the Segal-Bargmann space that we will encounter below during the quantisation of the Harmonic oscillator, see equation (6.3). The unitary transformation is the Segal-Bargman transform which we will define in Subsection 6.3.3.

4.4 The Schrödinger and Heisenberg Pictures

The Schrödinger and the Heisenberg pictures of Quantum Mechanics give a way of understanding how a quantum mechanical system evolves in time.

In the Schrödinger picture the time dependency of the system is encoded in the wave functions. The Hamilton operator of the quantum system $\widehat{H}$ is therefore assumed to be time-independent.

Definition 4.4.1. In the Schrödinger picture, the time evolution of the wave function ψ in a quantum system is given by the Schrödinger equation

$$i\hbar\frac{d\psi}{dt} = \widehat{H}\psi. \tag{4.1}$$

In the following, we will assume that we can find an orthonormal basis $\{\psi_j\}_{j\in\mathbb{N}}$ of the quantum Hilbert space consisting of eigenvectors of the Hamiltonian operator $\widehat{H}$. This is not necessarily true for self-adjoint operators and one should rather work with the spectral theorem to get generalised eigenspaces. More on that topic can be found in [4] Chapters 7 and 10. If we have an eigenbasis $\{\psi_j\}_{j\in\mathbb{N}}$ with corresponding eigenvalues E_j, the time evolution of these states is given by $\psi_j(t) = \exp(\frac{itE_j}{\hbar})\psi_j$, as one can verify by plugging $\psi_j(t)$ into the Schrödinger equation 4.1.
To find the basis of eigenvectors $\{\psi_j\}_{j\in\mathbb{N}}$, we will solve the following differential equation.

Definition 4.4.2. If $\widehat{H}$ is the Hamilton operator for a quantum system the eigenvector equation

$$\widehat{H}\psi = E\psi,\ E \in \mathbb{R}$$

is called the time-independent Schrödinger equation. A solution of this equation is called a stationary state.

Example 4.4.3. Consider the quantum system $L^2(\mathbb{R})$ with the Hamiltonian $\widehat{H} = \frac{\widehat{p}^2}{2m} + \frac{1}{2}m\omega^2\widehat{q}^2$, where m is the mass of the particle and ω is the angular frequency. Then the time-independent Schrödinger equation reads

$$\left(\frac{\widehat{p}^2}{2m} + \frac{1}{2}m\omega^2\widehat{q}^2\right)\psi = E\psi.$$

The stationary states are of the form

$$\psi_j(x) = \frac{1}{\sqrt{2^j j!}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\exp\left(\frac{-m\omega x^2}{2\hbar}\right)H_j\left(\sqrt{\frac{m\omega}{\hbar}}x\right),$$

where H_j are the Hermite polynomials given by

$$H_j(z) = (-1)^j e^{z^2}\frac{d^j}{dz^j}\left(e^{-z^2}\right).$$

The corresponding eigenvalues are given by $E_j = \hbar\omega(j + \frac{1}{2})$. We will quantise the Harmonic oscillator in Section 6.3 and we will find the same Hamiltonian operator (up to changing the coefficients of the position and momentum operators) in the Schrödinger position representation.

In the Heisenberg picture, the wave functions are assumed to be time independent and the time dependency of the system is encoded in the operators.

Definition 4.4.4. In the Heisenberg picture, a time dependent quantum observable $A = A(t)$ evolves in time according to the differential equation

$$\frac{dA(t)}{dt} = \frac{1}{i\hbar}[A(t), \widehat{H}],$$

where $\widehat{H}$ is the Hamiltonian of the system and where $[\cdot,\cdot]$ is the commutator given by $[A, B] = AB - BA$.

Chapter 5

Prequantisation

5.1 Dirac Conditions

Paul Dirac has stated conditions to quantise a classical system. These make sure that operators constructed during the quantisation process are valid quantum operators corresponding to classical observables.

Dirac Quantisation Conditions. *Let (X, ω, H) be a Hamiltonian system and $\mathcal{P}$ be a subalgebra of the Poisson algebra $(C^\infty(M), \{\cdot, \cdot\})$. Quantisation of the Hamiltonian system requires to construct a complex Hilbert space $\mathcal{H}$ and a map $q\colon \mathcal{P} \to \mathrm{End}(\mathcal{H})$ such that:*

1. *$q(1) = \lambda \mathrm{id}_{\mathcal{H}}$ for a suitable constant $\lambda \neq 0$,*
2. *$q(\{f, g\}) = c[q(f), q(g)]$, $f, g \in \mathcal{P}$, $c \in \mathbb{C}$,*
3. *all $q(f)$ are self-adjoint.*

Remark 5.1.1. The map q is going to transform classical observables into quantum operators. None of the Dirac conditions involve the use of a Hamiltonian. The second condition then means that the dynamics of the quantum operators (Heisenberg picture) is going to follow the same dynamics as the classical observables. This is where the Hamiltonian will be used.

5.2 Prequantisation Criteria

We now move on to prequantisation. The goal of this section is to explain what properties a classical system, i.e. a symplectic manifold, should have to be prequantisable and how to construct a prequantum line bundle.

We first discuss when a connection is said to be compatible with a Hermitian metric on a Hermitian vector bundle. We then discuss a topological property that closed compact surfaces have. Finally, we will see that there are two criteria that a line bundle should fulfil to be a prequantum bundle.

5.2.1 Compatible Connections

In this subsection, we will discuss connections compatible with Hermitian metrics. We will restrict the discussion to Hermitian line bundles $\pi\colon L \to M$ with Hermitian metric $h \in \Omega^0(M, (L \otimes \overline{L})^*)$. We will denote them with (L, h).

Definition 5.2.1. A connection ∇ on (L, h) is called compatible with h if for all sections $s, t \in \Omega^0(M, L)$ and all vector fields $X \in \Omega^0(M, TM)$, we have

$$X.h(s,t) = h(\nabla_X s, t) + h(s, \nabla_X t).$$

Proposition 5.2.2. A connection ∇ on L is compatible with a Hermitian metric h, if and only if the local gauge potentials $(\alpha_j)_{j\in J}$ with respects to local trivialisations $\varphi_j : L|_{U_j} \to U_j \times \mathbb{C}$ can be chosen to be purely imaginary 1-forms $\alpha_j \in i\Omega^1(M, \mathbb{R})$.

Proof. A system of local trivialisations (U_j, φ_j) can be chosen such that the φ_j are isomorphisms of Hermitian line bundles with respect to the constant Hermitian metric $h_0(\cdot,\cdot)$ from Example 2.4.46. Now we also know that local sections $s, t \in \Omega^0(U_j, L)$ have the form $s = fs_j,\ t = gs_j$ where $g, f \in C^\infty(U_j)$ and $s_j(x) = \varphi_j^{-1}(x, 1),\ x \in U_j$. Hence,

$$h_0(s,t) = h_0((x, f(x)), (x, g(x))) = \overline{f}(x)g(x)$$

Thus

$$X.h_0(s,t) = \left(X.\overline{f}\right) g + \overline{f}\,(X.g)$$

and

$$\begin{aligned} h_0(\nabla_X s, t) &= h_0((x, X.f(x) + \alpha_j|_X f(x)), (x, g(x))) \\ &= (\overline{X.f}(x) + \overline{\alpha}_j|_{\overline{X}}\overline{f}(x))g(x), \end{aligned}$$

$$h_0(s, \nabla_X t) = \overline{f}(x)(X.g(x) + \alpha_j|_X g(x)).$$

Therefore the compatibility condition is equivalent to

$$\left(X.\overline{f}\right) g + \overline{f}\,(X.g) = (\overline{X.f})g + \overline{f}X.g + (\overline{\alpha}_j(\overline{X}) + \alpha_j(X))\overline{f}g.$$

If we restrict to real vector fields we get the condition:

$$0 = \overline{\alpha}_j(X) + \alpha_j(X)$$

Thus α_j has to be a purely imaginary form. The converse is clear from the above equations. $\square$

Remark 5.2.3. From Proposition 5.2.2 we understand that the set of all connections compatible with the Hermitian metric h on L is the affine space modelled on $i\Omega^1(M, \mathbb{R})$. Additionally the curvature form $\Omega = \mathrm{Curv}(\nabla)$ of a connection on L compatible with a given Hermitian metric is always a purely imaginary two form $\Omega \in i\Omega^2(M, \mathbb{R})$.

5.2.2 Integrality of the Curvature

We will now discuss an integrality condition for the curvature form. First we will relate parallel transport to an integral over the curvature form.

Definition 5.2.4. We denote the set of all loops in M based at the point $x \in M$ with $\mathcal{L}(x)$, i.e. closed smooth curves which start and end in a fixed point $x \in M$.

Proposition 5.2.5. Let $S \subseteq M$ be an oriented compact surface embedded in M with boundary ∂S. Let $x \in \partial S$ a point in the boundary and $\gamma \in \mathcal{L}(x)$ a loop parameterising the boundary. The parallel transport $\widehat{\mathbb{P}}_\gamma \colon L_x \to L_x$ along γ is given by

$$\widehat{\mathbb{P}}_\gamma(l) = Q(\gamma)l,\ l \in L_x,\ Q(\gamma) = \exp\left(\int_S \Omega\right).$$

Proof. First we note that it is sensible that the parallel transport $\widehat{\mathbb{P}}_\gamma$ is given by a complex number $Q(\gamma) \in \mathbb{C}^\times$ such that $\widehat{\mathbb{P}}_\gamma : l \mapsto Q(\gamma)l$, since L_x is 1-dimensional. We now have to show that $Q(\gamma) = \exp(\int_S \Omega)$. It suffices to prove this result locally, hence we can assume the line bundle to be trivial. The horizontal lift of $\gamma \in M$ has the form $\lambda(t) = (\gamma(t), \xi(t)),\ t \in I := (t_0, t_1)$ and fulfils the condition from Definition 2.7.41. We know from Remark 2.7.46 that the second condition, i.e. that λ is horizontal, is equivalent to $\nabla_{\gamma'}\xi = 0$. More regorously, this is written as $(\gamma^*\nabla)_{\partial_t}(\xi) = 0$ [1], which yields a differential equation:

$$\begin{aligned}
0 &= (\gamma^*\nabla)_{\partial_t}(\xi)\\
&= (\gamma^*(d-\alpha))_{\partial_t}(\xi)\\
&= (d-\gamma^*\alpha)_{\partial_t}(\xi)\\
&=< d\xi, \partial_t > - < \gamma^*\alpha, \partial_t > \xi(t)\\
&= \xi'(t) - \alpha_{\gamma(t)}\left(D\gamma[\partial_t]\right)\xi(t)\\
&= \xi'(t) - \alpha_{\gamma(t)}(\gamma'(t))\xi(t)
\end{aligned}$$

A solution to this equation is given by:

$$\begin{aligned}
&\xi(t) = c\rho(t),\ c \in \mathbb{C},\\
&\rho(t) = \exp\left(\int_{t_0}^t \alpha_{\gamma(s)}(\gamma'(s))ds\right) = \exp\left(\int_I \gamma^*(\alpha)\right)
\end{aligned}$$

The constant c is given by the initial condition, i.e. $c = \xi(t_0)$. The integral can be written as

$$\int_I \gamma^*(\alpha) \overset{(1)}{=} \int_{\gamma(I)} \alpha = \int_{\partial S} \alpha \overset{(2)}{=} \int_S d\alpha \overset{(3)}{=} \int_S \Omega,$$

[1] We will not define the pullback connection in full generality, however in a local trivialisation the pullback connection is given by $\gamma^*(d-\alpha_j) = d - \gamma^*(\alpha_j)$.

where we used the definition of the integral for (1), Stokes theorem for (2) and Remark 2.8.3 for (3). Now $\widehat{\mathbb{P}}_\gamma(x,z) = (x, z\rho(t_1)) = \rho(t_1)(x,z) = Q(\gamma)(x,z)$, where $z = \xi(t_0)$, $l = (x,z) \in L_x$ and $Q(\gamma) = \rho(t_1) = \exp(\int_S \Omega)$.

□

Now with this proposition it is possible to show the following integrality condition.

Proposition 5.2.6. Let (L, ∇) be a line bundle with connection. Then the curvature form $\Omega = \mathrm{Curv}(\nabla)$ satisfies the integrality condition:

$$\int_\Sigma \Omega \in 2\pi i\mathbb{Z}$$

for every oriented closed surface $\Sigma \subseteq M$.

Recall that by Definition a closed manifold is a compact manifold without boundary.

Proof (Sketch). Let $(U_j, \varphi_j)_{j\in J}$ be the system of local trivialisations with local connection forms $\{\alpha_j\}_{j\in J}$. Let Σ be an oriented compact surface smoothly embedded into M. We can find a simple closed smooth curve γ dividing Σ into two parts S, S' such that S is an oriented compact surface with boundary ∂S parametrized by γ, S' is another compact surface with boundary $\partial S'$ parametrized by γ^-, and $S \cup S' = \Sigma$, $S \cap S' = \partial S = \partial S'$. Let us first assume that $\Sigma \subseteq U_j$ for some j. Fix $x \in \partial S$ and consider $\gamma \in \mathcal{L}(x)$, then the parallel transport along γ from Proposition 5.2.5 is given by

$$Q = \exp\left(\int_\gamma \alpha_j\right) = \exp\left(\int_S \Omega\right),$$

and the parallel transport along γ^- is given by

$$Q^- = \exp\left(\int_{\gamma^-} \alpha_j\right) = \exp\left(\int_{S'} \Omega\right).$$

Now from the first point of Definition 2.7.4, we know that $\widehat{\mathbb{P}}_\gamma^{-1} = \widehat{\mathbb{P}}_{\gamma^-}$. Therefore, the following holds

$$\begin{aligned} 1 = Q^-Q &= \exp\left(\int_{S'} \Omega\right) \exp\left(\int_S \Omega\right) \\ &= \exp\left(\left(\int_{S'} \Omega + \int_S \Omega\right)\right) = \exp\left(\int_\Sigma \Omega\right). \end{aligned}$$

Thus $\int_\Sigma \Omega \in 2\pi i\Sigma$. More generally we could partition Σ into pieces which are in suitable U_j's, i.e. $\Sigma = \bigcup_{j\in J} \Sigma \cap U_j$ and the following would still hold

$$Q = \exp\left(\int_S \Omega\right),\ Q^- = \exp\left(\int_{S'} \Omega\right).$$

□

5.2.3 Prequantisation Criteria

We finally discuss the prequantisation criteria.

Definition 5.2.7. A set of prequantum data for (M, ω) is a triple (L, h, ∇) consisting of a Hermitian line bundle (L, h) equipped with an h-compatible connection ∇ satisfying the prequantum condition $\mathrm{Curv}(\nabla) = -i\omega$. The line bundle L is called the prequantum line bundle and ∇ the prequantum connection.

Now we want to find conditions on (M, ω) such that prequantum data exists.

Definition 5.2.8. A differential 2-form β on the smooth manifold M is integral if $\int_\Sigma \beta \in \mathbb{Z}$ for all embedded closed surface $\Sigma \subseteq M$.

Definition 5.2.9. A symplectic manifold (M, ω) is said to be prequantisable if $\frac{\omega}{2\pi}$ is integral.

This definition is motivated by the following theorem.

Theorem 5.2.10. *A symplectic manifold (M, ω) is prequantisable if and only if it admits prequantum data. In other words there exists a Hermitian prequantum line bundle (L, h) and a prequantum connection ∇ if and only if $\frac{\omega}{2\pi}$ is integral.*

Remark 5.2.11. By Proposition 5.2.6, the integrality of $\omega/2\pi$ is necessary because, if Σ is an embedded closed surface:

$$\int_\Sigma \frac{\omega}{2\pi} = \int_\Sigma \frac{i\Omega}{2\pi} = \frac{i}{2\pi}\int_\Sigma \Omega \in \frac{i}{2\pi} 2\pi i\mathbb{Z} = \mathbb{Z}$$

We will not prove the other direction of the theorem. A proof can be given using Čech cohomology, which is explained in [2], Chapters 9 and 10.

Remark 5.2.12. To summarise, we will have to check the following two conditions to know if a Hermitian line bundle (L, h) with connection ∇ is a prequantum bundle for a symplectic form ω on M:

1. The curvature $\mathrm{Curv}(\nabla)$ satisfies the prequantisation condition
$$\mathrm{Curv}(\nabla) = -i\omega.$$
2. The connection ∇ is compatible with the Hermitian metric
$$X.h(s,t) = h(\nabla_X s, t) + h(s, \nabla_X t), \tag{5.1}$$
for all sections $s, t \in \Omega^0(M, L)$ and all vector fields $X \in \Omega^0(M, TM)$.

5.3 Constructing the Prequantum Hilbert Space

We will now discuss how to construct a prequantum Hilbert space out of sections on a prequantum line bundle and how to construct prequantum operators on the Hilbert space. We finally summarise what the prequantisation process is.

Proposition 5.3.1. Let (L, h, ∇) be a set of prequantum data for the $2n$-dimensional symplectic manifold (M, ω). The space $\widetilde{\mathcal{H}} := \{s \in \Omega^0(M, L) \mid \int_M h(s, s)\mu < \infty\}$ is an infinite dimensional $\mathbb{C}$-vector space equipped with the inner product $< \cdot \mid \cdot >$:

$$(s, t) \mapsto < s \mid t > := \int_M h(s, t)\mu,$$

where μ denotes the Liouville form $\frac{\omega^{\wedge n}}{n!}$.

Proof. It is clear that $\widetilde{\mathcal{H}}$ forms a vector space over $\mathbb{C}$ under the operation:

$$(s_1 + zs_2)(x) = s_1(x) + zs_2(x), \; s_1, s_2 \in \Omega^0(M, L), \; z \in \mathbb{C}, \; x \in M.$$

We can verify that $< \cdot \mid \cdot >$ is indeed an inner product, i.e. we have to show that it is sesquilinear, Hermitian and positive-definite. This follows directly from the fact that the Hermitian metric h is itself sesquilinear, Hermitian and positive-definite. □

Recall that a metric space is complete if every Cauchy sequence is convergent. The $\mathbb{C}$-vector space $\widetilde{\mathcal{H}}$ is a pre-Hilbert space, because it is not complete for the metric induced by the inner product:

$$(d(s, t))^2 = < s - t \mid s - t > . \tag{5.2}$$

We denote with $\mathcal{H}$ the completion of the pre-Hilbert space $\widetilde{\mathcal{H}}$ with respect to $d(\cdot, \cdot)$. This is the Hilbert space of the prequantisation of (M, ω) with respect to (L, h, ∇). We can then define prequantum operators that act on $\mathcal{H}$, as in the following proposition.

Proposition 5.3.2. For any classical observable $f \in C^\infty(M)$ we define an operator

$$\widehat{f} = \mu_f + i\nabla_{X_f},$$

where μ_f is the multiplication from the left with the function f. The assignment $q(f) := \widehat{f}$ satisfies the first two Dirac quantisation conditions from Section 5.1.

Remark 5.3.3. Note that the operator $\widehat{f}$ is only going to be defined on some suitable domain $D \subseteq \mathcal{H}$. Indeed, the multiplication by f need not satisfy square integrability and ∇_X is only defined on smooth sections. We will however not determine the biggest domain D where it is defined.

Proof. The first criterion is clear with $\lambda = 1$.
For the second one, let $f, g \in C^\infty(M)$ be classical observables. The prequantisation criterion tells us that the curvature will be equal to $-i\omega$, hence:

$$[\nabla_{X_f}, \nabla_{X_g}] - \nabla_{[X_f, X_g]} = \mathrm{Curv}(\nabla) = -i\omega(X_f, X_g).$$

Using the fact that $\omega(X_f, X_g) = \{f, g\}$ and that $[X_f, X_g] = -X_{\{f,g\}}$ we get:

$$[\nabla_{X_f}, \nabla_{X_g}] = -\nabla_{X_{\{f,g\}}} - i\{f, g\}. \tag{5.3}$$

The second equation that will be needed in the calculation is the following. Let s be any section of the prequantum line bundle, then we find:

$$\begin{aligned}[\mu_f, \nabla_{X_g}]s &= \mu_f \nabla_{X_g} s - \nabla_{X_g}(fs) \\ &= f\nabla_{X_g} s - (X_g.f)s - f\nabla_{X_g} s \\ &= -(X_g.f)s = -\{f, g\}s,\end{aligned} \tag{5.4}$$

where we have used the Leibniz rule and that $X_g.f = \{f, g\}$. Analogously we find

$$[\nabla_{X_f}, \mu_g] = -\{f, g\}. \tag{5.5}$$

We now test if the second Dirac condition is fulfilled:

$$\begin{aligned}[\widehat{f}, \widehat{g}] &= [\mu_f + i\nabla_{X_f}, \mu_g + i\nabla_{X_g}] \\ &= (i)^2[\nabla_{X_f}, \nabla_{X_g}] + [\mu_f, \mu_g] + i([\mu_f, \nabla_{X_g}] + [\nabla_{X_f}, \mu_g]) \\ &\overset{(1)}{=} -(-\nabla_{X_{\{f,g\}}} - i\{f, g\}) - 2i(\{f, g\}) \\ &= \nabla_{X_{\{f,g\}}} - i\{f, g\} \\ &= (-i)(\mu_{\{f,g\}} + i\nabla_{X_{\{f,g\}}}) \\ &= (-i)\widehat{\{f, g\}},\end{aligned}$$

where we have used equation 5.3, 5.4 and 5.5 in (1). We have checked that the second Dirac quantisation $[\widehat{f}, \widehat{g}] = c\widehat{\{f, g\}}$ is fulfilled with $c = -i$. □

Proposition 5.3.4. The operator $\widehat{f}$ from Proposition 5.3.2 is symmetric, i.e.:

$$< \widehat{f}s_1 \mid s_2 >=< s_1 \mid \widehat{f}s_2 >,$$

where s_1, s_2 are defined on suitable domains.

Remark 5.3.5. We haven't discussed the Lie derivative in the Differential geometry background Chapter as we have (and will not) use it much for the quantisation purpose. We will however use it in the proof of Proposition 5.3.4. We therefore define the Lie derivative for differential forms and give without proof a few of its properties. Let $k \in \mathbb{N}$. First, we define the interior product $\iota_X : \Omega^k(M) \to \Omega^{k-1}(M)$ by setting $(\iota_X\omega)(X_1, ..., X_{k-1}) := \omega(X, X_1, ..., X_{k-1})$ for $X, X_1, ..., X_{k-1} \in \Omega^0(M, TM)$. The interior product fulfils the following property for $\alpha \in \Omega^k(M),\ \beta \in \Omega^l(M),\ k, l \in \mathbb{N}$ and $X \in \Omega^0(M, TM)$:

$$\iota_X(\alpha \wedge \beta) := \iota_X\alpha \wedge \beta + (-1)^k\alpha \wedge \iota_X\beta. \tag{5.6}$$

We can now define the Lie derivative $\mathcal{L}_X : \Omega^k(M, TM) \to \Omega^k(M, TM)$ with respect to a vector field X via the Cartan formula:

$$\mathcal{L}_X = d \circ \iota_X + \iota_X \circ d. \tag{5.7}$$

The Lie derivative has the following properties:

1. $\mathcal{L}_X(\alpha \wedge \beta) = \mathcal{L}_X\alpha \wedge \beta + \alpha \wedge \mathcal{L}_X\beta$ (Leibniz)
2. $\mathcal{L}_X(\alpha) = X.\alpha$ for $\alpha \in \Omega^0(M) \cong C^\infty(M)$

Proof (Proposition 5.3.4). We first want to show that

$$(X_f.h(s_1, s_2))\mu = \mathcal{L}_{X_f}(h(s_1, s_2)\mu), \tag{5.8}$$

where $\mu = \frac{\omega^{\wedge n}}{n!}$ is the Liouville measure and $h(\cdot, \cdot)$ is the Hermitian metric on L. We compute:

$$\begin{aligned}\mathcal{L}_{X_f}(h(s_1, s_2)\mu) &\overset{(1)}{=} \mathcal{L}_{X_f}(h(s_1, s_2))\mu + h(s_1, s_2)\mathcal{L}_{X_f}\mu \\ &\overset{(2)}{=} (X_f.h(s_1, s_2))\mu + h(s_1, s_2)\mathcal{L}_{X_f}\mu,\end{aligned}$$

where we have used the first property of the Lie derivative from Remark 5.3.5 for (1) and the second property for (2). We see that we now only have to show that $\mathcal{L}_{X_f}\mu$ vanishes. Before doing so, we calculate the following:

$$\begin{aligned}\iota_{X_f}\omega^{\wedge n} &\overset{(1)}{=} (\iota_{X_f}\omega) \wedge \omega^{\wedge n-1} + (-1)^2\omega \wedge (\iota_{X_f}\omega^{\wedge n-1}) \\ &\overset{(2)}{=} df\omega \wedge \omega^{\wedge n-1} + \omega \wedge df \wedge \omega^{\wedge n-2} + \cdots + \omega^{\wedge n-1} \wedge df \\ &\overset{(3)}{=} ndf \wedge \omega^{\wedge n-1},\end{aligned} \tag{5.9}$$

where we have used equation (5.6) for (1), the fact that $\iota_{X_f}\omega = \omega(X_f, \cdot) = df$ according to Remark 3.0.4 for (2) and Remark 2.5.26 for (3). We can now compute:

$$\begin{aligned}\mathcal{L}_{X_f}\mu &\overset{(1)}{=} d(\iota_{X_f}\mu) + \iota_{X_f}(d\mu) \\ &\overset{(2)}{=} d(\iota_{X_f}\mu) \\ &= \frac{d(\iota_{X_f}\omega^{\wedge n})}{n!} \\ &\overset{(3)}{=} d\left(\frac{df \wedge \omega^{\wedge n-1}}{(n-1)!}\right) \\ &\overset{(4)}{=} 0,\end{aligned}$$

where we have used Cartan formula (5.7) in (1), Remark 2.5.31 in (2), equation (5.9) for (3) and Remark 2.5.31 again as well as $d^2 = 0$ for (4).

Finally we can prove the desired result. We compute:

$$
\begin{aligned}
< \widehat{f}s_1 \mid s_2 > &= \int_M h(\widehat{f}s_1, s_2)\mu \\
&= \int_M h(fs_1 + i\nabla_{X_f}s_1, s_2)\mu \\
&\overset{(1)}{=} \int_M iX_f.h(s_1, s_2)\mu + \int_M h(s_1, i\nabla_{X_f}s_2)\mu + h(s_1, fs_2)\mu \\
&\overset{(2)}{=} \int_M i\mathcal{L}_{X_f}(h(s_1, s_2)\mu) + \int_M h(s_1, \widehat{f}s_2)\mu \\
&\overset{(3)}{=} i\left(\int_M d \circ \iota_{X_f}(h(s_1, s_2)\mu) + \int_M \iota_{X_f} \circ d(h(s_1, s_2)\mu)\right) + \int_M h(s_1, \widehat{f}s_2)\mu \\
&\overset{(4)}{=} \int_M h(s_1, \widehat{f}s_2)\mu \\
&=< s_1 \mid \widehat{f}s_2 >,
\end{aligned}
$$

where we have used the compatibility condition from Definition 5.2.2 for (1), the result (5.8) for (2), Cartan formula (5.7) in (3) and Stokes' Theorem as well as Remark 2.5.31 for (4).

□

Remark 5.3.6. In the finite dimensional case, a symmetric operator as in Proposition 5.3.4 is a self-adjoint operator. However, this is not the case on an infinite dimensional vector space. Recall from section 4.1, that on a Hilbert space, an operator is self adjoint if it has the same domain of definition as its adjoint operator and if it coincides with its adjoint on its domain of definition. Since $\widehat{f}$ is an unbounded operator on a Hilbert space, the fact that $\widehat{f}$ is symmetric doesn't imply that it is self-adjoint. The interested reader can find more on that topic in [4].

Nevertheless if for a smooth function $f \in C^\infty(M)$ its Hamiltonian vector field $X_f \in \Omega^0(M, L)$ is complete on M, one can show that the operator $\widehat{f}$ is a well-defined self-adjoint operator:

$$\widehat{f} \colon \mathcal{H} \to \mathcal{H}.$$

Thus all the Dirac conditions are fulfilled for such an operator, i.e. $\widehat{f}$ is a quantum operator.

The proof that $\widehat{f}$ as above is self-adjoint is based on Stone's theorem, i.e. one should show that $\widehat{f}$ is the infinitesimal generator of a strongly continuous one-parameter unitary group on $\mathcal{H}$ (see Proposition 9.5 in [2] and Theorem 10.15 in [4]).

Remark 5.3.7. To summarise, the prequantisation process is the following. We have to construct a Hermitian line bundle with connection ∇ on a symplectic manifold M such that the two conditions from Remark 5.2.12 are satisfied. We then have to build the pre-hilbert space $\widetilde{\mathcal{H}}$ out of sections of the prequantum

line bundle (L, h, ∇) and complete it with respect to the metric (5.2). We can then construct prequantum operators on the prequantum Hilbert space which correspond to classical observables as in Proposition 5.3.2.
The constructed prequantum Hilbert space is however too big. Indeed, some of the sections of the prequantum line bundle depend on the whole phase space. In particular, some of them depend on position and momentum simultaneously which is forbidden by Heisenberg uncertainty principle. We will therefore have to introduce the concept of polarisation (see next Chapter). We are going to build our quantum Hilbert space out of polarised sections of the prequantum Hilbert space.

5.4 The 2-dimensional Flat Example - Prequantisation

We will now proceed with the prequantisation of the trivial line bundle $\pi : L \to M$ on the 2-dimensional symplectic manifold $(M = \mathbb{R}^2, \omega)$ from Example 2.6.5. As a remainder, we define $M = T^*\mathbb{R}$ with global canonical coordinates q and p. The symplectic form is given by $\omega = dq \wedge dp$ and there is a symplectic potential $\alpha := -pdq$ such that $d\alpha = \omega$. For the trivial line bundle $L := M \times \mathbb{C} \to M$, we can take the Hermitian metric to be the constant one, i.e. $h_0 \colon L \times L \to \mathbb{C}$ and $h_0((l_1, \lambda_1), (l_2, \lambda_2)) := \lambda_1 \overline{\lambda}_2$. We know from Remark 2.7.6 that any connection on L has the form $\nabla = d - \beta$ where $\beta \in \Omega^1(M, \mathbb{C})$ is a global connection form. We first check that ω fulfils the integrality condition from Remark 5.2.12. We integrate ω on an arbitrary oriented, closed, compact surface $\Sigma \subseteq M$:

$$\int_\Sigma \omega = \int_\Sigma d\alpha \overset{(1)}{=} \int_{\partial\Sigma} \alpha = 0,$$

where we used Stoke's Theorem for (1) and that Σ is closed. From Theorem 5.2.10, this is enough to make sure that there exists a set of prequantum data for (M, ω). We now check that this is indeed true for a good choice of ∇. Consider $\nabla := d - i\alpha$ then

$$-i\omega \overset{!}{=} \mathrm{Curv}(\nabla) = \Omega = -d(i\alpha) + (i\alpha) \wedge (i\alpha) \overset{(1)}{=} -id\alpha, \tag{5.10}$$

where we have used Remark 2.8.3 and in (1) the wedge product for $\mathbb{C}$-valued forms which is alternating. Therefore the choice $\nabla = d - i\alpha$ fulfils the first condition of Remark 5.2.12. It is also compatible with the Hermitian metric h_0 since $\beta = i\alpha$ is purely imaginary (recall Proposition 5.2.2).

We conclude that the trivial Hermitian line bundle $(M \times \mathbb{C}, h_0)$ with connection $\nabla = d - i\alpha$ on the symplectic manifold $(T^*\mathbb{R}, dq \wedge dp)$ is a prequantum line bundle.

We can now construct the prequantum Hilbert space out of sections of the prequantum line bundle, i.e.:

$$\widetilde{\mathcal{H}} := \left\{ s \in \Omega^0(M, L) \;\middle|\; \int_M h_0(s, s)\mu < \infty \right\}$$

with inner product

$$(s,t) \mapsto < s \mid t > := \int_M h_0(s,t)\mu,$$

where $\mu = \omega = dq \wedge dp$ is the Liouville form. We can then complete it with respect to the metric (5.2) induced by the inner product which yields the prequantum Hilbert space $\mathcal{H}$.

The operators $\widehat{f}$ are given by Proposition 5.3.6. We now calculate explicitly the prequantum operators for $f = \{q,p\}$. We know from subsection 3.0.3 that the Hamiltonian vector field corresponding to the classical observables q and p are given by $X_q = -\frac{\partial}{\partial p}$ and $X_p = \frac{\partial}{\partial q}$. Thus, using Remark 2.7.7 and the explicit form for the prequantum connection ∇, we find:

$$\begin{aligned}
\widehat{q} &= q + i\nabla_{X_q} \\
&= q + i(X_q - i\alpha|_{X_q}) \\
&= q - i\frac{\partial}{\partial p} + (-pdq)\left(-\frac{\partial}{\partial p}\right) \\
&= q - i\frac{\partial}{\partial p} \\
\widehat{p} &= p + i\nabla_{X_p} \\
&= p + i(X_p - i\alpha|_{X_p}) \\
&= p + i\frac{\partial}{\partial q} + (-pdq)\left(\frac{\partial}{\partial q}\right) \\
&= p + i\frac{\partial}{\partial q} - p \\
&= i\frac{\partial}{\partial q}.
\end{aligned}$$

For completeness, we check that the operators of position and momentum fulfil the second Dirac condition. For any $s \in \mathcal{H}$, we compute:

$$\begin{aligned}
[\widehat{q},\widehat{p}]s &= [q - i\frac{\partial}{\partial p}, i\frac{\partial}{\partial q}]s \\
&= iq\frac{\partial}{\partial q}s + \frac{\partial}{\partial p}\frac{\partial}{\partial q}s - i\frac{\partial}{\partial q}(qs) - \frac{\partial}{\partial q}\frac{\partial}{\partial p}s \\
&\overset{(1)}{=} iq\frac{\partial}{\partial q}s - is - iq\frac{\partial}{\partial q}s \\
&= -is \\
&= -i\widehat{1}s \\
&= -i\widehat{(dq \wedge dp)\left(-\tfrac{\partial}{\partial p}, \tfrac{\partial}{\partial q}\right)} \\
&= -i\widehat{\omega\left(-\tfrac{\partial}{\partial p}, \tfrac{\partial}{\partial q}\right)}s \\
&= -i\widehat{\{q,p\}}s,
\end{aligned}$$

where we have used the symmetry of the second derivative and Leibniz for partial derivatives in (1).

We see that the position operator has a dependency on both q and p. We would like instead to have position and momentum operators which are only dependent on either q or p, for example

$$\widehat{q} = q,\ \widehat{p} = i\frac{\partial}{\partial q},$$

the usual position and momentum operators in quantum mechanics in the position space representation (up to a constant). This shows explicitly why we have to choose a polarisation, which we will discuss in the next chapter.

Chapter 6

Quantisation

6.1 Polarisations

We will now define polarisations. We will first define real polarisations and then complex polarisations. We finally discuss how one should use them to construct the quantum Hilbert space and we continue our example of the flat case. Throughout this section (M, ω) is a symplectic manifold of dimension $2n$.

6.1.1 Real Polarisations

We will first define Lagrangian vector spaces and we will then generalise this notion to manifolds. For the following discussion, let (V, ω) be a symplectic vector space and let $W \subseteq V$ be a subspace of V.

Definition 6.1.1. We define the symplectic orthogonal of W as

$$W^{\perp_\omega} := \{v \in V \mid \omega(v, w) = 0, \ \forall w \in W\}.$$

The following proposition follows from the non-degeneracy of the symplectic form.

Proposition 6.1.2. The dimension formula

$$\dim W + \dim W^{\perp_\omega} = \dim V$$

holds for $W^{\perp_\omega}$ the symplectic orthogonal of W.

Remark 6.1.3. We note however that the intersection of W with its symplectic orthogonal need not be empty, i.e. $W \cap W^{\perp_\omega} \neq \{0\}$ in general. The following definitions are therefore sensible.

Definition 6.1.4. The subspace $W \subseteq V$ is

- isotropic if $W \subseteq W^{\perp_\omega}$,

- Lagrangian if $W = W^{\perp_\omega}$.

We can then state the following Corollary to Proposition 6.1.2.

Corollary 6.1.5. If W is isotropic then $\dim W \leq \frac{1}{2}\dim V$. If W is Lagrangian then $\dim W = \frac{1}{2}\dim V$.

Remark 6.1.6. We note that Lagrangian can also be defined as maximally isotropic.

We can now define real polarisations.

Definition 6.1.7. A real polarisation on M is an integrable Lagrangian distribution $P \subseteq TM$. In other words, a real polarisation $P \subseteq TM$ is a Lagrangian foliation (recall Definition 2.2.57).

Remark 6.1.8. Explicitly, the condition for a foliation $P \subseteq TM$ to be a real polarisation is the following: $\omega_x(X, Y) = 0,\ x \in M,\ X, Y \in P_x$ and no larger subspace of T_xM which contains T_xM properly has this property (it is maximal isotropic).

Proposition 6.1.9. A distribution $P \subseteq TM$ is a real polarisation if and only if for each $x \in M$ there exists an open neighbourhood U of x and n smooth functions $f_1, ..., f_n \in C^\infty(U)$ such that:

1. The family of differentials $\{df_1, ..., df_n\}$ is a linearly independent family of functions,
2. for each $x \in U$, $P_x = \mathrm{span}_{\mathbb{R}}\{X_{f_1}(x), ..., X_{f_n}(x)\}$,
3. $\{f_j, f_k\} = 0,\ j, k \in \{1, ..., n\}$.

Proof. Let P be a real polarisation. Since P is by definition an integrable distribution, for every $x \in M$ there exists an integral manifold (L, ι), i.e. a leaf. From the implicit function theorem (see [1]) we can find n independent smooth functions $f_1, ..., f_n \in C^\infty(M)$ such that the leaves of P are locally of the form $\{x \in U \mid f_1(x) = c_1, f_2(x) = c_2, ..., f_n(x) = c_n\}$ with suitable constants $c_1, ..., c_n \in \mathbb{R}$ and $U \subseteq M$ an open subset. For each vector field $X \in \Omega^0(U, P)$, we have $X.f_i = 0, i = 1, ..., n$, hence

$$\omega(X_{f_i}, X) = df_i(X) = X.f_i = 0.$$

Thus $X_{f_i} \in \Omega^0(U, P)$, since P is Lagrangian, and $X_{f_1}, ..., X_{f_n}$ spans P locally. Therefore we also have $0 = \omega(X_{f_i}, X_{f_j}) = \{f_i, f_j\}$.
Conversely, the condition $0 = \{f_i, f_j\} = \omega(X_{f_i}, X_{f_j})$ implies that P is isotropic. Since the f_i are linearly independent, the dimension of the distribution is indeed n and the leaves are given by $\{x \in U \mid f_1(x) = c_1, f_2(x) = c_2, ..., f_n(x) = c_n\}$. □

Let us now construct a real polarisation on a $2n$-dimensional flat symplectic manifold (if $n = 1$ we have the same setup as in Subsection 5.4).

Example 6.1.10. Let $(T^*\mathbb{R}^n, \omega)$ be a $2n$-dimensional symplectic manifold with global coordinates $\{q_1, ..., q_n, p_1, ..., p_n\}$ and symplectic form given by $\omega = \sum_k dq_k \wedge dp_k$. We define the vertical distribution $P \subseteq T(T^*\mathbb{R})$ as

$$P \coloneqq \text{span}_{\mathbb{R}} \left\{ \frac{\partial}{\partial p_i} \,\middle|\, 1 \leq i \leq n \right\}.$$

The globally defined n independent smooth functions $q_1, ..., q_n \in C^\infty(M)$ are in Poisson involution, that is they generate a commutative Poisson subalgebra. The span of their Hamiltonian vector fields gives P, indeed:

$$P = \text{span}_{\mathbb{R}} \left\{ \frac{\partial}{\partial p_i} \,\middle|\, 1 \leq i \leq n \right\} = \text{span}_{\mathbb{R}} \left\{ X_{q_i} \mid 1 \leq i \leq n \right\},$$

since we know from subsection 3.0.3 that the Hamiltonian vector field corresponding to the classical observables q_k is given by $X_{q_k} = -\frac{\partial}{\partial p_k}$. Therefore by Proposition 6.1.9, P is a real polarisation and the leaves of the polarisation are given by

$$\{(q_1, ..., q_n, p_1, ..., p_n) \in T^*\mathbb{R}^n \mid q_i = c_i,\ c_i \in \mathbb{C}\}.$$

With a similar argument, one can show that the horizontal distribution given by

$$P \coloneqq \text{span}_{\mathbb{R}} \left\{ \frac{\partial}{\partial q_i} \,\middle|\, 1 \leq i \leq n \right\} = \text{span}_{\mathbb{R}} \left\{ X_{p_i} \mid 1 \leq i \leq n \right\}$$

is a real polarisation.

We see that there exists a real polarisation for our flat example, however we should note that a real polarisation need not exist in general. We therefore need complex polarisations.

6.1.2 Complex Polarisations

We first need to define complexified tangent bundles which will be needed in the following discussion.

Definition 6.1.11. Let V be a real vector space. The complexification of the vector space V is given by:

$$V^{\mathbb{C}} \coloneqq V \otimes_{\mathbb{R}} \mathbb{C}$$

and the scalar multiplication is given by

$$\lambda \cdot (v \otimes \mu) = v \otimes \lambda\mu,\ \mu, \lambda \in \mathbb{C},\ v \in V.$$

Remark 6.1.12. The complexified vector space $V \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic to $V \oplus iV \coloneqq \{v_1 + iv_2 \mid v_1, v_2 \in V\}$. Indeed the map $V \otimes_{\mathbb{R}} \mathbb{C} \to V \oplus iV,\ v \otimes 1 \mapsto v$ is a canonical isomorphism of $\mathbb{C}$-vector spaces.

Remark 6.1.13. Since the complexification and the isomorphism from 6.1.12 do not involve a choice of basis, the Metatheorem implies that the same construction holds for vector bundles and in particular for the tangent bundle TM. Thus, its complexification $TM^{\mathbb{C}}$ is the vector bundle with fibres at $x \in M$ given by $(TM^{\mathbb{C}})_x := (T_xM)^{\mathbb{C}} = T_xM \otimes_{\mathbb{R}} \mathbb{C} \cong T_xM \oplus iT_xM$.

Definition 6.1.14. Let V be a real vector space and $V^{\mathbb{C}}$ its complexification. There exists an $\mathbb{R}$-linear involution $\bar{\cdot}\colon V^{\mathbb{C}} \to V^{\mathbb{C}}$ defined by

$$v \otimes \lambda \mapsto v \otimes \overline{\lambda},\ v \otimes \lambda \in V^{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C},$$

or equivalently with $v_1 + iv_2 \in V^{\mathbb{C}} = V \oplus iV$:

$$v_1 + iv_2 \mapsto v_1 - iv_2.$$

Remark 6.1.15. This involution is defined without a choice of basis. From the Metatheorem we can use it on vector bundle and specifically on tangent bundle. Therefore for a distribution $P \subseteq TM$ we can use the involution fiberwise and thus $\overline{P}_x$ is defined within $T_xM^{\mathbb{C}}$.

Definition 6.1.16. Let (V, ω) be a real symplectic vector space. Its complexification $V^{\mathbb{C}}$ carries a complex symplectic structure $\omega^{\mathbb{C}}\colon V^{\mathbb{C}} \times V^{\mathbb{C}} \to \mathbb{C}$ defined by:

$$\omega^{\mathbb{C}}(v \otimes \lambda, w \otimes \mu) := \omega(v, w)\lambda\mu,\ v \otimes \lambda,\ w \otimes \mu \in V^{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C},$$

or equivalently for $v_1 + iv_2,\ w_1 + iw_2 \in V \oplus iV$:

$$\omega^{\mathbb{C}}(v_1 + iv_2, w_1 + iw_2) := \omega(v_1, w_1) - \omega(v_2, w_2) + i(\omega(v_1, w_2) + \omega(v_2, w_1)).$$

Remark 6.1.17. In the complexification, non-degeneracy for $\omega^{\mathbb{C}}$ means that for $v \in V$, the map

$$\begin{aligned} V^{\mathbb{C}} &\to \left(V^{\mathbb{C}}\right)^* \\ v &\mapsto \omega^{\mathbb{C}}_v = \omega^{\mathbb{C}}(v, \cdot) \end{aligned}$$

is an isomorphism.

Remark 6.1.18. From the Metatheorem, this can again be extended fiberwise on $TM^{\mathbb{C}}$. We can thus make sense of isotropic and Lagrangian distributions $P \subseteq TM^{\mathbb{C}}$, using the complex symplectic form.

Definition 6.1.19. Let (M, ω) be a symplectic manifold. A complex polarisation P of (M, ω) is a foliation $P \subseteq TM^{\mathbb{C}}$ such that

- $P_x \subseteq T_xM^{\mathbb{C}}$ is Lagrangian for all $x \in M$
- the distribution $D_x := P_x \cap \overline{P}_x \cap T_xM$ has constant rank.

The complex polarisation P is said to be real if $D_x = T_xM$ for all $x \in M$ and pseudo-Kähler (or purely complex) if $D_x = \{0\}$ for all $x \in M$.

Remark 6.1.20. Equivalently we could define a pseudo-Kähler polarisation as being a complex polarisation such that the Hermitian form $h\colon P \times P \to \mathbb{C},\ (X,Y) \mapsto h(X,Y) = -i\omega^{\mathbb{C}}(X,\overline{Y})$ is a non-degenerate Hermitian form. If further, h is positive definite we say that the polarisation is Kähler. The condition for the conjugate polarisation to be Kähler is that the Hermitian form $\overline{h}\colon \overline{P} \times \overline{P} \to \mathbb{C},\ (X,Y) \mapsto \overline{h}(X,Y) = i\omega^{\mathbb{C}}(X,\overline{Y})$ is a non-degenerate positive definite Hermitian form.

Example 6.1.21. Let us construct the Kähler polarisation for the same symplectic manifold as in Example 6.1.10, i.e. $M = T^*\mathbb{R}^n$ with symplectic form $\omega = \sum_k dq_k \wedge dp_k$. We can define a holomorphic atlas on $T^*\mathbb{R}^n$ using a single global chart $\varphi\colon T^*\mathbb{R}^n \to \mathbb{C}^n,\ (q_1,...,q_n,p_1,...,p_n) \mapsto (z_1,...,z_n)$, where $z_j := \frac{1}{\sqrt{2}}(q_j + ip_j)$ and we define P as the pointwise span of $\left(\frac{\partial}{\partial z_j}\right)_{j\in J}$, where $\frac{\partial}{\partial z_j} := \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial q_j} - i\frac{\partial}{\partial p_j}\right)$ and $J = \{1,...,n\}$. The conjugate distribution $\overline{P}$ is the pointwise span of $\left(\frac{\partial}{\partial \overline{z}_j}\right)_{j\in J}$, where $\frac{\partial}{\partial \overline{z}_j} := \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial q_j} + i\frac{\partial}{\partial p_j}\right)$.
By Frobenius theorem 2.2.56, the distribution $\overline{P} \subseteq TM^{\mathbb{C}}$ defines a foliation because it is involutive and has constant rank. Indeed, we compute with $Y = \sum_j y_j \frac{\partial}{\partial \overline{z}_j} \in \overline{P}$ and $W = \sum_j w_j \frac{\partial}{\partial \overline{z}_j} \in \overline{P}$ as well as a smooth function $f \in C^\infty(M)$

$$\begin{aligned}[Y,W]f &= \sum_{jk} y_j \frac{\partial}{\partial \overline{z}_j}\left(w_k \frac{\partial}{\partial \overline{z}_k} f\right) - w_k \frac{\partial}{\partial \overline{z}_k}\left(y_j \frac{\partial}{\partial \overline{z}_j} f\right)\\ &= \sum_{jk} y_j \frac{\partial w_k}{\partial \overline{z}_j}\frac{\partial}{\partial \overline{z}_k} f - w_k \frac{\partial y_j}{\partial \overline{z}_k}\frac{\partial}{\partial \overline{z}_j} f\\ &= \sum_{jk}\left(y_j \frac{\partial w_k}{\partial \overline{z}_j} - w_j \frac{\partial y_k}{\partial \overline{z}_j}\right)\frac{\partial}{\partial \overline{z}_k} f.\end{aligned}$$

Thus $[Y,W] \in \overline{P}$.
Now we want to show that it is Lagrangian. The 1-forms $dz_k \in \Omega^1(M,\mathbb{C}) = \Omega^0(M,(TM^{\mathbb{C}})^*)$ are defined as the dual to $\left(\frac{\partial}{\partial z_j}\right)_{j\in J}$, i.e. $dz_k\left(\frac{\partial}{\partial z_j}\right) := \delta_{kj}$ and $dz_k\left(\frac{\partial}{\partial \overline{z}_j}\right) := 0$. Similarly $d\overline{z}_k$ are defined via $d\overline{z}_k\left(\frac{\partial}{\partial z_j}\right) := 0$ and $d\overline{z}_k\left(\frac{\partial}{\partial \overline{z}_j}\right) := \delta_{kj}$. We compute the following:

$$dz_j \wedge d\overline{z}_j = d\left(\frac{q_j + ip_j}{\sqrt{2}}\right)\wedge d\left(\frac{q_j - ip_j}{\sqrt{2}}\right) = -\frac{i}{2}dq_j\wedge dp_j + \frac{i}{2}dp_j \wedge dq_j = -i dq_j \wedge dp_j.$$

Thus the formula for the $\mathbb{C}$-bilinear extension of the symplectic form is $\omega^{\mathbb{C}} \in \Omega^2(M,\mathbb{C}) = \Omega^0(M,\Lambda^2(TM^{\mathbb{C}})^*)$ as $\omega^{\mathbb{C}} = i\sum_j dz_j \wedge d\overline{z}_j$. We can now input elements of the foliation $\overline{P}$ which yield:

$$\omega^{\mathbb{C}}\left(\frac{\partial}{\partial \overline{z}_j}, \frac{\partial}{\partial \overline{z}_l}\right) = i\sum_k dz_k \wedge d\overline{z}_k\left(\frac{\partial}{\partial \overline{z}_j}, \frac{\partial}{\partial \overline{z}_l}\right) = 0.$$

Finally, the complex polarisation $\overline{P}$ is pseudo-Kähler. Indeed, we prove with the following calculation that $TM^{\mathbb{C}} = P \oplus \overline{P}$, which implies $P \cap \overline{P} = \{0\}$ for dimensions reasons. Let X be an element of $TM^{\mathbb{C}}$. From Remark 6.1.13, X can be uniquely written as a sum $X = Y + iZ$, where $X, Y \in TM$. Let us write Y and Z in global coordinates as $Y = \sum_j a_j \frac{\partial}{\partial q_j} + b_j \frac{\partial}{\partial p_j}$ and $Z = \sum_j c_j \frac{\partial}{\partial q_j} + d_j \frac{\partial}{\partial p_j}$. We can then write the sum as:

$$\begin{aligned} X &= Y + iZ \\ &= \sum_j a_j \frac{\partial}{\partial q_j} + b_j \frac{\partial}{\partial p_j} + \sum_k c_k \frac{\partial}{\partial q_k} + d_k \frac{\partial}{\partial p_k} \\ &= \sum_j (a_j + ic_j) \frac{\partial}{\partial q_j} + (b_j + id_j) \frac{\partial}{\partial p_j} \\ &= \sum_j \frac{a_j + ic_j}{\sqrt{2}} \left(\frac{\partial}{\partial z_j} + \frac{\partial}{\partial \overline{z}_j} \right) + i \frac{b_j + id_j}{\sqrt{2}} \left(\frac{\partial}{\partial z_j} - \frac{\partial}{\partial \overline{z}_j} \right). \end{aligned}$$

This shows that any $X \in TM$ can be uniquely written as a sum of elements of P and $\overline{P}$, since the coefficients a_j, b_j, c_j, d_j were unique. The polarisation is even Kähler, because

$$\begin{aligned} \overline{h}(\frac{\partial}{\partial \overline{z}_j}, \frac{\partial}{\partial \overline{z}_j}) &= i\omega^{\mathbb{C}}(\frac{\partial}{\partial \overline{z}_j}, \frac{\partial}{\partial z_j}) \\ &= -\sum_k dz_k \wedge d\overline{z}_k(\frac{\partial}{\partial \overline{z}_j}, \frac{\partial}{\partial z_j}) \\ &= dz_j(\frac{\partial}{\partial z_j}) d\overline{z}_j(\frac{\partial}{\partial \overline{z}_j}) \geq 0. \end{aligned}$$

An alternative definition of a Kähler polarisation is via Kähler manifold. A Kähler manifold then naturally induces a Kähler polarisation. To define such a manifold, we first need to define an almost complex structure.

Definition 6.1.22. A complex structure on a real vector space V is a linear transformation $J \colon V \to V$ such that $J^2 = -\mathrm{Id}$.

Remark 6.1.23. Necessarily the dimension of V is even, since

$$\det(J)^2 = \det(J^2) = \det(-\mathrm{Id}) = (-1)^{\dim(V)}.$$

Example 6.1.24. We can find a complex structure on $\mathbb{R}^{2n}$ by using the isomorphism

$$\begin{aligned} \mathbb{R}^{2n} &\to \mathbb{C}^n \\ (x_j, y_j)_{j \in I} &\mapsto (x_j + iy_j)_{j \in I}, \end{aligned}$$

where $I = \{1, ..., n\}$. Now the complex structure on $\mathbb{C}^n$ is given by the map $J \colon \mathbb{C}^n \to \mathbb{C}^n$, $v \mapsto iv$, i.e. $(x_j + iy_j)_{j \in I} \mapsto (-y_j + ix_j)_{j \in I}$. Thus the matrix of

J in the canonical basis of $\mathbb{R}^{2n}$ is

$$J = \begin{pmatrix} 0 & -\mathrm{Id}_{\mathbb{R}^n} \\ \mathrm{Id}_{\mathbb{R}^n} & 0 \end{pmatrix},$$

and it clearly holds that $J^2 = -\mathrm{Id}_{\mathbb{R}^{2n}}$.

Definition 6.1.25. An almost complex structure J on a manifold M is an almost complex structure on each fibre T_xM which varies smoothly with $x \in M$, i.e. $J \in \Omega^0(M, TM \otimes T^*M),\ J^2 = -\mathrm{Id}_{TM}$.

Remark 6.1.26. We note that any Kähler Polarisation induces an almost complex structure on (M, ω). Indeed, any element $v \in T_xM$ can be written as $v = Z(v) + \overline{Z}(v)$ where $Z(v) \in P$, since $T_xM^{\mathbb{C}} = P \oplus \overline{P}$. Therefore the map defined by $Jv \coloneqq iZ(v) - i\overline{Z}(v)$ is $\mathbb{R}$-linear and satisfies $J^2 = -\mathrm{Id}$. We will see later that J endows (M, ω) with a structure of a Kähler manifold, defined below.

Remark 6.1.27. An almost complex structure is said to be integrable if around every point $x \in M$ there exists local holomorphic coordinates which glued together gives a holomorphic atlas endowing M with a complex structure which induces the almost complex structure, i.e. in local holomorphic coordinates $z_j = x_j + iy_j, j \in \{1, ..., n\}$ the following holds:

$$J\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j},\ J\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j}.$$

The Newlander–Nirenberg theorem states that an almost complex structure J is integrable if and only if for all vector fields T_xM:

$$N_J(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y] = 0.$$

We can now define a Kähler manifold.

Definition 6.1.28. A Kähler manifold is a manifold M with a symplectic structure ω and an integrable almost complex structure J such that ω and J are compatible in the following sense:

$$g(X, Y) \coloneqq \omega(X, JY),\ X, Y \in \Omega^0(M, TM)$$

is a non-degenerate, positive-definite, symmetric tensor. We denote a Kähler manifold with the triple (M, ω, J).

Remark 6.1.29. The fact that the symplectic form ω and the almost complex structure J are compatible as in Definition 6.1.28 implies that J is a linear symplectic transformation, i.e.

$$\omega(JX, JY) = \omega(X, Y),\ X, Y \in \Omega^0(M, TM).$$

Indeed, we can compute the following:

$$\omega(JX, JY) = g(JX, Y) \stackrel{(1)}{=} g(Y, JX) = \omega(Y, J^2X) = \omega(X, Y),$$

where we have used that g is symmetric in (1).

Proposition 6.1.30. Suppose that (M, ω, J) is a Kähler manifold, then the eigenbundles of J are Kähler polarisations. In other words, there exists a natural Kähler polarisation P defined pointwise as the eigenspace of $J_x : T_x M^{\mathbb{C}} \to T_x M^{\mathbb{C}}$ corresponding to the eigenvalue i and the conjugate polarisation $\overline{P}$ is given by the eigenspace corresponding to the eigenvalue $-i$.

Proof (Sketch). It automatically follows that $P \cap \overline{P} = \{0\}$, since P and $\overline{P}$ are eigenbundles corresponding to different eigenvalues. Let us check that the distribution P is involutive. Using Remark 6.1.27, We compute for $X, Y \in P$:

$$0 = N_J(X, Y) = -[X, Y] - iJ[X, Y] - iJ[X, Y] - [X, Y].$$

Rearranging this equation yields

$$J[X, Y] = i[X, Y]$$

which shows that P is involutive.
Finally we verify if P is Lagrangian. This follows from the dimension of P and the following calculation, for all $X, Y \in P$:

$$\omega(X, Y) = \omega(JX, JY) = \omega(iX, iY) = -\omega(X, Y),$$

thus $\omega(X, Y) = 0$. □

6.1.3 The Quantisation Process

Now that we understand polarisations, we can finalise our discussion about quantisation.

Definition 6.1.31. Given a polarisation on a symplectic manifold (M, ω), the polarised functions are the functions $f \in C^\infty(M)$ with

$$X.f = 0$$

for all $X \in \Omega^0(M, P)$. We denote the space of all polarised functions as $C_P^\infty(M)$.

Remark 6.1.32. We note that $X.f = 0$ implies that $[X_f, X] \in \Omega^0(M, P)$. Indeed we calculate for all $X \in \Omega^0(M, P)$:

$$0 = X.f = df(X) \overset{(1)}{=} \omega(X_f, X),$$

where we have used Remark 3.0.4 in (1). This calculation shows that X_f is also an element of $\Omega^0(M, P)$ since P is Lagrangian. Because P is additionally involutive, it follows that $[X_f, X] \in \Omega^0(M, P)$ for all $X \in \Omega^0(M, P)$.

Definition 6.1.33. Given a polarisation on a symplectic manifold (M, ω), the polarised sections in a line bundle $\pi : L \to M$ with connection ∇ are the sections $s \in \Omega^0(M, L)$ with

$$\nabla_X s = 0$$

for all $X \in \Omega^0(M, P)$. We denote the space of all polarised sections as $\Omega_P^0(M, L)$.

Given a prequantum Hilbert space $\mathcal{H}$ as constructed in the prequantisation process (see Remark 5.3.7), we can define a suitable polarisation P on the symplectic manifold M. We then construct the quantum Hilbert space $\mathcal{H}_P$ as the polarised sections of the prequantum Hilbert space, i.e. $\mathcal{H}_P$ is the completion of

$$\tilde{\mathcal{H}}_P := \mathcal{H} \cap \Omega^0_P(M, L).$$

By doing this we can avoid violating the Heisenberg uncertainty principle as the chosen polarisation will enable us to rule out the sections which are dependent on both position and momentum, as we will make explicit in the flat 2-dimensional example. The next propositions will show that polarised classical observables induce quantum operators on the quantum Hilbert space $\mathcal{H}_P$.

Proposition 6.1.34. The space of polarised functions is closed under the Poisson bracket, i.e. $\{f, g\} \in C^\infty_P(M)$ for any $f, g \in C^\infty_P(M)$.

Proof. With $X \in \Omega^0(M, P)$ and $f, g \in C^\infty_P(M)$, we show that the Poisson bracket $\{f, g\}$ is in $C^\infty_P(M)$. We compute:

$$\begin{aligned} X.\{f, g\} &\overset{(1)}{=} X.(X_g.f) \\ &\overset{(2)}{=} X.(X_g.f) - X_g.(X.f) + X_g.(X.f) \\ &\overset{(3)}{=} [X, X_g].f \overset{(4)}{=} 0, \end{aligned}$$

where we have used equation (3.5) in (1), we have added 0 in (2), we have used that $X.f = 0$ for $f \in C^\infty_P(M)$ in (3) and Remark 6.1.32 in (4). □

Proposition 6.1.35. For any polarised classical observable $f \in C^\infty_P(M)$, the prequantum operator of Proposition 5.3.2 acts on the space of smooth functions of the prequantum line bundle preserving polarised ones.

Proof. From Proposition 5.3.2 and Proposition 5.3.4, we already know that $\widehat{f}$ restricted to the polarised sections fulfils the first two Dirac conditions and that it is symmetric. We only need to show that $\nabla_X \widehat{f} s = 0$ for $X \in \Omega^0(M, P)$, $f \in C^\infty_P(M)$ and $s \in \Omega^0_P(M, L)$. First, let us note that the condition that the curvature equals $-i\omega$ is equivalent to:

$$-i\omega(X, X_f)s = [\nabla_X, \nabla_{X_f}]s - \nabla_{[X, X_f]}s,$$

where we have used the definition of the curvature from Definition 2.8.2. Rearranging the terms and using the fact that $s \in \Omega^0_P(M, L)$ as well as $[X, X_f] \in \Omega^0(M, P)$ yields:

$$\begin{aligned} \nabla_X(\nabla_{X_f}s) &= -i\omega(X, X_f)s + \nabla_{X_f}(\nabla_X s) + \nabla_{[X, X_f]}s \\ &= -i\omega(X, X_f)s. \end{aligned}$$

We also have from Remark 3.0.4 that

$$-i\omega(X, X_f)s = i\omega(X_f, X)s = i(X.f)s,$$

and thus we find:

$$\nabla_X(\nabla_{X_f} s) = i(X.f)s. \tag{6.1}$$

With this result we can now compute the following:

$$\begin{aligned}
\nabla_X(\widehat{f}s) &= \nabla_X(fs + i\nabla_{X_f} s) \\
&\overset{(1)}{=} (X.f)s + f\nabla_X s + i\nabla_X(\nabla_{X_f} s) \\
&\overset{(2)}{=} (X.f)s - (X.f)s = 0,
\end{aligned}$$

where we have used Leibniz in (1) and equation 6.1 in (2). □

Remark 6.1.36. We have now finally constructed the desired quantum Hilbert space out of square integrable polarised sections equipped with quantum operators which preserve polarised sections (e.g. prequantum operators associated to polarised functions). However in doing so a few non-trivial issues should be discussed in general. The space of polarised sections could be empty, because there could be no polarised sections which are square sumable. A way around this issue is to define a measure on M/D (which however need not even be Hausdorff) and then redefine the quantum Hilbert space as sections on M/P with inner product defined by integrating against this measure. We will show this in the 2-dimensional case. In general, to overcome these difficulties one could introduce half densities and distributional sections. The interested reader can find more on that topic in [7] and [2].

6.2 The 2-dimensional Flat Example - Quantisation

We now continue our discussion on the 2-dimensional flat symplectic manifold from Example 5.4. We will use real polarisation in this example and Kähler polarisations later on to quantise the n-dimensional Harmonic oscillator.
First recall from Example 6.1.10 that we can define the vertical polarisation on $T^*\mathbb{R}$ as the distribution defined by

$$P = \mathrm{span}_{\mathbb{R}}\{\frac{\partial}{\partial p}\}.$$

We can now use this polarisation to define our quantum Hilbert space. The polarised section s of the prequantum line bundle are given by $s = fs_1$, where $f \in C^\infty(M)$ and s_1 is the constant section $s_1(x) = (x, 1)$, which satisfy:

$$\nabla_{\frac{\partial}{\partial p}} s = 0.$$

Since we already know the form of the prequantum connection $\nabla = d - i\alpha = d + ipdq$ we can compute:

$$\begin{aligned}\nabla_{\frac{\partial}{\partial p}} fs_1 &= \frac{\partial f}{\partial p}s_1 + ipdq(\frac{\partial}{\partial p})fs_1 \\ &= \frac{\partial f}{\partial p}s_1\end{aligned}$$

Thus our quantum Hilbert space $\mathcal{H}_P$ will be composed of the sections with coefficient function satisfying the following differential equation:

$$\frac{\partial f}{\partial p} = 0. \tag{6.2}$$

In words, equation (6.2) means that the sections in our Hilbert space cannot have any explicit dependency on momentum, i.e. the coefficient functions are $f(p,q) = f(q)$.
The classical observable q induces a quantum operator $\widehat{q}$ from Proposition 6.1.35. Indeed, q belongs to $C^\infty_P(M)$ since it is clear that $\frac{\partial}{\partial p}(q) = 0$. Now we already know that $\widehat{q} = q - i\frac{\partial}{\partial p}$ from Example 5.4. Thus if we restrict the operator $\widehat{q}$ to $\mathcal{H}_p$, $i\frac{\partial}{\partial p}$ will vanish as equation (6.2) states. Therefore we end up with the quantum operator $\widehat{q} = q$ as desired.
Even though the classical observable p doesn't belong to $C^\infty_P(M)$ as $\frac{\partial}{\partial p}(p) = 1$, it also induces a quantum operator $\widehat{p}$. Indeed, we already know from Example 5.4 that its corresponding quantum operator $\widehat{p}$ is given by $\widehat{p} = i\frac{\partial}{\partial q}$ which preserves polarised sections, since for any $fs_1 \in \mathcal{H}_P$

$$\nabla_{\frac{\partial}{\partial p}}(\widehat{p}(fs_1)) = \nabla_{\frac{\partial}{\partial p}}\left(i\frac{\partial f}{\partial q}s_1\right) = \frac{\partial}{\partial p}\left(i\frac{\partial f}{\partial q}\right)s_1 - pdq\left(\frac{\partial}{\partial p}\right)\frac{\partial f}{\partial q}s_1 \overset{(1)}{=} 0$$

where we have used the symmetry of the second derivative and $dq\left(\frac{\partial}{\partial p}\right) = 0$ in (1). These two operators fulfil the commutation relation $[\widehat{q},\widehat{p}] = -i$, which is the canonical commutation relation from Quantum mechanics up to a constant.

We now have the desired polarised sections of the prequantum line bundle and the correct quantum operators for position and momentum which preserve polarised sections. However, we should note that the Hilbert space $\mathcal{H}_P$ is actually empty, since for any element fs_1 of $\mathcal{H}_P$ the integral:

$$\int_M h_0(fs_1, fs_1)dq \wedge dp = \int_{\mathbb{R}^2} f(q)f^*(q)dqdp$$

diverges as the integral over p is unbounded. Therefore, we have to redefine the Hilbert space as sections over the space of leaves M/P and then look for a suitable measure on this space against which we will integrate. We already know from Example 6.1.10 that the leaves are given by $\{(c,p) \mid c \in \mathbb{C}\}$. The subspace

$$Q := \mathbb{R} \oplus \{0\} = \{(q,0) \mid q \in \mathbb{R}\} \subseteq T^*\mathbb{R}$$

is a global transverse to the leaves and Q is diffeomorphic to M/P. We can redefine our quantum Hilbert space out of sections whose coefficient functions are elements of $C^\infty_P(Q,\mathbb{C}) = C^\infty(Q,\mathbb{C})$. This is sensible because any section $s \in \Omega^0_P(M,L) \cong C^\infty_P(M,\mathbb{C})$ is uniquely determined by $s|_Q$, its restriction on Q. In other words the map

$$\begin{gathered} C^\infty_P(M,\mathbb{C}) \to C^\infty(Q,\mathbb{C}) \\ s \mapsto s|_Q \end{gathered}$$

is a diffeomorphism of $\mathbb{C}$-vector spaces. Indeed, the map is injective, since $s(q,p)$ is uniquely defined by the initial value problem:

$$\begin{cases} \frac{\partial s}{\partial p} = 0 \\ s(q,0) = s|_Q(q), \end{cases}$$

because of the Picard–Lindelöf theorem. A measure on $M/P \cong Q$ is given by the Lebesque measure dq. We can now construct explicitly the following Hilbert space:

$$\tilde{\mathcal{H}}'_P := \left\{ f \in C^\infty(Q,\mathbb{C}) \;\middle|\; \int_Q f^* f dq < \infty \right\} = C^\infty(Q,\mathbb{C}) \cap L^2(Q,\mathbb{C}).$$

Now by completing it we finally end up with the desired quantum Hilbert space:

$$\mathcal{H}'_P = L^2(Q,\mathbb{C}),$$

the Schrödinger position representation, with the operators $\widehat{q} = q$ and $\widehat{p} = i\frac{\partial}{\partial q}$ which fulfil the canonical commutation relation.

We note that we could choose similarly the horizontal polarisation:

$$P := \mathrm{span}_{\mathbb{R}}\{\frac{\partial}{\partial q}\}.$$

We would find for the quantum operator of momentum $\widehat{p} = p$ and for the quantum operator of position $\widehat{q} = i\frac{\partial}{\partial p}$ on the quantum Hilbert space $\mathcal{H}_P$. The space of leaves would be $\{(0,p) \mid p \in \mathbb{R}\}$ and the measure on this space would be the Lebesque measure dp. It would then yield the quantum Hilbert space corresponding to the Schrödinger momentum representation. We recall that the Fourier transformation isomorphically maps the position representation to the momentum representation.

6.3 The n-dimensional Harmonic Oscillator

We will now fully describe the quantisation of the n-dimensional harmonic oscillator. First let us discuss the real polarisation case and then the Kähler polarisation case.

6.3.1 The Real Polarisation Case

We start with a flat symplectic manifold i.e. $M = T^*\mathbb{R}^n$ with symplectic form $\omega = \sum_{j=1}^n dq_j \wedge dp_j$ in Darboux coordinates q_i, p_i. The symplectic potential is given by the 1-form $\alpha = -\sum_{j=1}^n p_j dq_j$. In the real case, the quantum Hilbert space with respect to the vertical polarisation $P = \mathrm{span}_{\mathbb{R}}\{\frac{\partial}{\partial p_1}, \cdots, \frac{\partial}{\partial p_n}\}$ is the same as the one in Example 6.2 generalised to the $2n$-dimensions, i.e. $\mathcal{H}'_P = L^2(Q, \mathbb{C})$ where $Q = \mathbb{R}^n \oplus \{0\}$. The classical Hamiltonian operator is given by $H = \frac{\|p\|^2}{2m} + \frac{1}{2}m\omega^2\|q\|^2$ where m is the mass of the particle, ω is the angular frequency and $\|\cdot\|$ is the n-dimensional Euclidian norm. The constants have a physical meaning which is irrelevant in the process of quantisation. We can therefore set $m = \hbar = \omega = 1$ for convenience. We write $H = \frac{1}{2}(\|p\|^2 + \|q\|^2)$. Using equation (3.6) generalised to $2n$-dimension, the Hamiltonian vector field to the Hamiltonian H takes the form:

$$X_H = \sum_j \frac{\partial H}{\partial p_j}\frac{\partial}{\partial q_j} - \frac{\partial H}{\partial q_j}\frac{\partial}{\partial p_j} = \sum_j p_j \frac{\partial}{\partial q_j} - q_j \frac{\partial}{\partial p_j}.$$

We can now construct the prequantum operator analogously to what was done for the position and momentum operators in Example 5.4. We calculate:

$$\begin{aligned}
\widehat{H} &= H + i\nabla_{X_H} \\
&= H + i(X_H - i\alpha|_{X_H}) \\
&= H - i\left(\sum_j p_j\frac{\partial}{\partial q_j} - q_j\frac{\partial}{\partial p_j}\right) + \left(-\sum_k p_k dq_k\right)\left(\sum_j p_j\frac{\partial}{\partial q_j} - q_j\frac{\partial}{\partial p_j}\right) \\
&= \sum_j \frac{1}{2}p_j^2 + \frac{1}{2}q_j^2 - i\left(p_j\frac{\partial}{\partial q_j} - q_j\frac{\partial}{\partial p_j}\right) - p_j^2 \\
&= \sum_j -\frac{1}{2}p_j^2 - ip_j\frac{\partial}{\partial q_j} + \frac{1}{2}q_j^2 + iq_j\frac{\partial}{\partial p_j}.
\end{aligned}$$

We note however that this operator is not a quantum operator, because $\widehat{H}$ doesn't preserve polarised sections. Indeed, $\widehat{H}f$ doesn't fulfil the condition 6.2 as we check in the following, for any $f \in \mathcal{H}'_P$:

$$\begin{aligned}
\frac{\partial}{\partial p_l}(\widehat{H}f) &= \frac{\partial}{\partial p_l}\left(\sum_j -\frac{1}{2}p_j^2 - ip_j\frac{\partial}{\partial q_j} + \frac{1}{2}q_j^2 + iq_j\frac{\partial}{\partial p_j}\right)f \\
&\overset{(1)}{=} -p_l f - ip_l\frac{\partial^2 f}{\partial p_l q_j} \\
&\overset{(2)}{=} -p_l f,
\end{aligned}$$

where we have used that every term of the form $\frac{\partial f}{\partial p_k}$ for some $k \in \mathbb{N}$ vanishes in (1) and we have used the symmetry of the partial derivatives in (2). The same problem arises with a different choice of symplectic potential or with a different choice of polarisation. To overcome this issue, we will use Kähler polarisation and after solving the harmonic oscillator on the complexified prequantum bundle, we will come back to the space $\mathcal{H}'_P$ with the help of the Segal-Bargmann transform.

6.3.2 The Kähler Polarisation Case

Let us first discuss the $(n = 1)$-dimensional case. We will then generalise to any $n \in \mathbb{N}$ dimension. We start again with a flat symplectic manifold i.e. $M = T^*\mathbb{R}$ with symplectic form $\omega = dq \wedge dp$ in global coordinates q, p. The Hamiltonian of the 1-dimensional Harmonic oscillator is given by $H = \frac{1}{2}(p^2 + q^2)$.
As in Example 6.1.21, we can define the holomorphic coordinates $z = \frac{1}{\sqrt{2}}(q+ip)$ and we can write the symplectic form in these coordinates as:

$$\omega^{\mathbb{C}} = idz \wedge d\overline{z}$$

and it is clear that the 1-form

$$\alpha^{\mathbb{C}} = \frac{i}{2}(zd\overline{z} - \overline{z}dz)$$

is a potential of $\omega^{\mathbb{C}}$. Since $\frac{1}{\sqrt{2}}(q + ip)\frac{1}{\sqrt{2}}(q - ip) = \frac{1}{2}(q^2 + p^2)$, the Hamiltonian of the Harmonic oscillator takes the form

$$H^{\mathbb{C}} = z\overline{z}.$$

In the following, we will write ω, α and H, neglecting the index $\mathbb{C}$.
Also from Example 6.1.21, we know that

$$\overline{P} := \operatorname{span}_{\mathbb{C}}\left(\frac{\partial}{\partial\overline{z}}\right)$$

defines a Kähler polarisation.
The construction of the prequantum Hilbert space is analogous to what we have done in Example 5.4. Indeed, the trivial line bundle $L := M \times \mathbb{C}$ equipped with the constant Hermitian metric h_0 and connection $\nabla = d - i\alpha$ is a prequantum line bundle. The prequantum Hilbert space is then given by the completion of

$$\widetilde{\mathcal{H}} := \left\{ s \in \Omega^0(M, L) \;\middle|\; \int_M h_0(s, s)\mu < \infty \right\},$$

where $\mu = \omega = idz \wedge d\overline{z}$ is the Liouville form.
This space is however too big and we therefore need to restrict to polarised sections of the prequantum line bundle. We already know that the sections $s \in \Omega^0(M, L)$ of the trivial line bundle are given by coefficient functions $f \in C^\infty(M)$ and the constant section $s_1 \in \Omega^0(M, L)$, i.e. $s(x) = f(x)s_1(x) = f(x)(x, 1) = (x, f(x)),\ x \in M$. The polarised sections are therefore the ones which satisfy:

$$\nabla_{\frac{\partial}{\partial\overline{z}}} f s_1 = \left(\frac{\partial}{\partial\overline{z}} f - i\alpha|_{\frac{\partial}{\partial\overline{z}}} f\right) s_1 = 0$$

We can write this condition as:

$$\frac{\partial f}{\partial \overline{z}} - i\left(\frac{i}{2}(zd\overline{z} - \overline{z}dz)\right)\left(\frac{\partial}{\partial \overline{z}}\right) f = \frac{\partial f}{\partial \overline{z}} + \frac{1}{2} z f = 0.$$

We can solve this differential equation with separation of variable:

$$\frac{df}{f} = -\frac{1}{2} z d\overline{z} \;\Leftrightarrow\; \log(f) = -\frac{z\overline{z}}{2} + k,$$

for some branch of the logarithm and k some holomorphic function, since it needs to fulfil $\frac{\partial k}{\partial \overline{z}} = 0$. Since the exponential function is holomorphic and the composition of holomorphic functions is again holomorphic, we understand that the coefficient functions need to be of the form:

$$f = \varphi \exp\left(-\frac{|z|^2}{2}\right),$$

where $\varphi \in \mathcal{O}_M$ is a holomorphic function, i.e. $\frac{\partial \varphi}{\partial \overline{z}} = 0$. Thus the polarised sections $s \in \Omega^0_P(M, L)$ are of the form $s = \varphi s_e$ where $\varphi \in \mathcal{O}_M$ and $s_e := \exp\left(-\frac{|z|^2}{2}\right) s_1$. Thus there is an obvious isomorphism between the space of polarised section $\Omega^0_P(M, L)$ and $\mathcal{O}_M$ which is given by $\varphi s_e \mapsto \varphi$, since s_e is a canonical choice of global nowhere-vanishing polarised section. The inner product on $\Omega^0(M, L)$ induces an inner product on the space of holomorphic functions $\mathcal{O}_M$. Indeed, let $fs_e, gs_e \in \Omega^0_P(M, L)$ be two polarised sections, we can then compute:

$$\begin{aligned}
< fs_e \mid gs_e > &= \int_M h_0(fs_e, gs_e) dq \wedge dp \\
&= \int_M f\overline{g} h_0(s_e, s_e) dq \wedge dp \\
&= \int_M f\overline{g} \exp\left(-|z|^2\right) dq \wedge dp.
\end{aligned}$$

We can therefore define on $\mathcal{O}_M$ the following inner product $(\cdot \mid \cdot)$ induced by the inner product $< \cdot \mid \cdot >$:

$$(f \mid g) = c \int_{\mathbb{C}} f\overline{g} \exp\left(-|z|^2\right) i dz \wedge d\overline{z},$$

where $c \in \mathbb{R}$ is some multiplicative constant. We fix $c = \frac{1}{2\pi}$ such that $(1 \mid 1) = 1$, indeed:

$$\begin{aligned}
c \int_{\mathbb{C}} \exp\left(-\frac{q^2 + p^2}{2}\right) dq \wedge dp &= \frac{1}{2\pi}\left(\int_{\mathbb{R}} \exp\left(-\frac{q^2}{2}\right) dq\right)\left(\int_{\mathbb{R}} \exp\left(-\frac{p^2}{2}\right) dp\right) \\
&= \frac{1}{2\pi}(\sqrt{2\pi})^2 = 1.
\end{aligned}$$

Hence, we can identify the quantum Hilbert space as the space of holomorphic functions which are square integrable with respect to the measure $\exp\left(-|z|^2\right) dz \wedge d\overline{z}$, i.e.:

$$\mathcal{H}_P := \left\{ \varphi \in \mathcal{O}_M \;\middle|\; \frac{i}{2\pi} \int_{\mathbb{C}} |\varphi|^2 \exp(-|z|^2) dz \wedge d\overline{z} < \infty \right\}. \tag{6.3}$$

This space is called the Segal-Bargmann space. This space is already complete, since it is closed in $L^2(M, L)$, the completion of the space of square integrable sections $\Omega^0(M, L)$. The proof of this fact can be found in [4]. Because the Segal-Bargmann space is closed, we can write $L^2(M, L) = \mathcal{H}_P \oplus (\mathcal{H}_P)^{\perp}$, with respect to the inner product $< \cdot \mid \cdot >$. We can therefore define a projection map $\pi_{\mathcal{O}} \colon L^2(M, L) \to \mathcal{H}_P$, which we will use below.
We now construct the operators on $\mathcal{H}_P$ corresponding to z, $\overline{z}$ and the Hamiltonian of the 1-dimensional harmonic oscillator. We first need to construct the Hamiltonian vector field X_C associated to any classical observable C. The matrix representation of the symplectic form in the complex frame is given by:

$$\omega = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

Using equation (3.6) and the matrix of ω, we can write:

$$X_C = i \left(\frac{\partial C}{\partial z} \frac{\partial}{\partial \overline{z}} - \frac{\partial C}{\partial \overline{z}} \frac{\partial}{\partial z} \right)$$

Thus for z and $\overline{z}$, we get:

$$\begin{aligned} X_z &= i \frac{\partial}{\partial \overline{z}} \\ X_{\overline{z}} &= -i \frac{\partial}{\partial z}, \end{aligned}$$

and the Hamiltonian vector field corresponding to H is:

$$X_H = i \left(\frac{\partial H}{\partial z} \frac{\partial}{\partial \overline{z}} - \frac{\partial H}{\partial \overline{z}} \frac{\partial}{\partial z} \right) = i \left(\overline{z} \frac{\partial}{\partial \overline{z}} - z \frac{\partial}{\partial z} \right).$$

We can now use Proposition 5.3.2 to build prequantum operators. We note that from Proposition 6.1.35 $\widehat{z}$ will be a well-defined operator on $\mathcal{H}_P$, since it is clear that $\frac{\partial}{\partial \overline{z}} z = 0$. We calculate the prequantum operator $\widehat{z}$:

$$\begin{aligned} \widehat{z} &= z + i \nabla_{i \frac{\partial}{\partial \overline{z}}} \\ &= z + i \left(i \frac{\partial}{\partial \overline{z}} - i \left(\frac{i}{2} (z d\overline{z} - \overline{z} dz) \left(i \frac{\partial}{\partial \overline{z}} \right) \right) \right) \\ &= z + i \left(i \frac{\partial}{\partial \overline{z}} + i \frac{z}{2} \right) \\ &= \frac{z}{2} - \frac{\partial}{\partial \overline{z}}, \end{aligned}$$

and the prequantum operator $\widehat{\overline{z}}$:

$$\begin{aligned}
\widehat{\overline{z}} &= \overline{z} + i\nabla_{-i\frac{\partial}{\partial z}} \\
&= \overline{z} + i\left(-i\frac{\partial}{\partial z} - i\left(\frac{i}{2}(zd\overline{z} - \overline{z}dz)\left(-i\frac{\partial}{\partial z}\right)\right)\right) \\
&= \overline{z} + \frac{\partial}{\partial z} - \frac{\overline{z}}{2} \\
&= \frac{\overline{z}}{2} + \frac{\partial}{\partial z}.
\end{aligned}$$

We see that restricting the operator $\widehat{z}$ on $\mathcal{H}_P$ yields $\widehat{z} = \frac{z}{2}$, since every function in $\mathcal{H}_P$ is holomorphic. We see that $\widehat{z}$ clearly preserves $\mathcal{H}_P$ and is thus a well-defined operator on $\mathcal{H}_P$. The operator $\widehat{\overline{z}}$ does not preserve $\mathcal{H}_P$, since $\frac{\partial \overline{z}}{\partial \overline{z}} = 1 \neq 0$, i.e. multiplication by $\overline{z}$ does not preserve polarised sections. Hence, we will compose $\overline{z}$ with the projection $\pi_{\mathcal{O}}$ to get a well-defined operator. We decompose $\overline{z}\varphi$ uniquely as $\pi_{\mathcal{O}}(\overline{z}\varphi) + \psi$, where $\varphi \in \mathcal{H}_P$ and $\psi \in (\mathcal{H}_P)^{\perp}$, and we compute the following, for some $g \in \mathcal{H}_P$:

$$\begin{aligned}
(\overline{z}\varphi \mid g) &= \frac{1}{2\pi}\int_{\mathbb{C}} \varphi\overline{gz}\exp(-|z|^2)dq \wedge dp \\
&\overset{(1)}{=} -\frac{1}{2\pi}\int_{\mathbb{C}} \varphi\overline{g}\frac{\partial}{\partial z}\exp(-|z|^2)dq \wedge dp \\
&\overset{(2)}{=} \frac{1}{2\pi}\int_{\mathbb{C}} \frac{\partial \varphi}{\partial z}\overline{g}\exp(-|z|^2)dq \wedge dp - \frac{1}{2\pi}\int_{\mathbb{C}} \frac{\partial}{\partial z}(\varphi\overline{g}\exp(-|z|^2))dq \wedge dp \\
&\overset{(3)}{=} \frac{1}{2\pi}\int_{\mathbb{C}} \frac{\partial \varphi}{\partial z}\overline{g}\exp(-|z|^2)dq \wedge dp \\
&= \left(\frac{\partial}{\partial z}\varphi \mid g\right).
\end{aligned} \tag{6.4}$$

We have used in (1) that $\overline{z}\exp(-|z|^2) = -\frac{\partial}{\partial z}\exp(-|z|^2)$. We have then used in (2) that $\frac{\partial}{\partial z}(\varphi\overline{g}\exp(-|z|^2)) = \varphi\overline{g}\frac{\partial}{\partial z}\exp(-|z|^2) + \frac{\partial \varphi}{\partial z}\overline{g}\exp(-|z|^2)$, since $\frac{\partial}{\partial z}\overline{g} = 0$ as g is holomorphic. Finally, we can compute the following

$$\begin{aligned}
\frac{\partial}{\partial z}(\varphi\overline{g}\exp(-|z|^2))dq \wedge dp &= \frac{1}{\sqrt{2}}\frac{\partial}{\partial q}(f\overline{g}\exp(-|z|^2))dq \wedge dp \\
&\quad - \frac{i}{\sqrt{2}}\frac{\partial}{\partial p}(f\overline{g}\exp(-|z|^2))dq \wedge dp \\
&= d\left(\frac{1}{\sqrt{2}}(f\overline{g}\exp(-|z|^2))dp\right) \\
&\quad + d\left(\frac{i}{\sqrt{2}}(f\overline{g}\exp(-|z|^2))dq\right),
\end{aligned}$$

as $dp \wedge dp = 0$ and $d(dp) = 0$. With this calculation it follows from Stokes' theorem that the second term vanishes in (3). Now we understand from what we have computed in (6.4) that for all $g \in \mathcal{H}_P$

$$(\pi_{\mathcal{O}}(\overline{z}\varphi) \mid g) = (\overline{z}\varphi \mid g) = \left(\frac{\partial}{\partial z}\varphi \;\middle|\; g \right),$$

which is equivalent to

$$\left(\pi_{\mathcal{O}}(\overline{z}\varphi) - \frac{\partial}{\partial z} \;\middle|\; g \right) = 0.$$

Since $(\cdot \mid \cdot)$ is positive definite and non-degenerate, it implies that

$$\frac{\partial \varphi}{\partial z} - \pi_{\mathcal{O}}(\overline{z}\varphi) = 0.$$

Hence, the component of $\widehat{\overline{z}}$ taking values in $\mathcal{H}_P$ reads:

$$\widehat{\overline{z}} = \frac{\partial}{\partial z} + \frac{1}{2}\pi_{\mathcal{O}} \circ \overline{z} = \frac{3}{2}\frac{\partial}{\partial z}.$$

This operator is a well-defined operator on $\mathcal{H}_P$ since it preserves the Hilbert space $\mathcal{H}_P$. Indeed, we compute the following for completion, with $\varphi \in \mathcal{H}_P$:

$$\frac{\partial}{\partial \overline{z}}(\widehat{\overline{z}}\varphi) = \frac{\partial}{\partial \overline{z}}(\frac{3}{2}\frac{\partial}{\partial z}\varphi) = \frac{3}{2}\frac{\partial^2 \varphi}{\partial z \partial \overline{z}} = 0,$$

where we have used the symmetry of the partial derivatives.
We now move on to the prequantum Hamiltonian operator, which is given by:

$$\begin{aligned}
\widehat{H} &= H + i\nabla_{X_H} \\
&= H + i(X_H - i\alpha|_{X_H}) \\
&= z\overline{z} + i(X_H - i\left(\frac{i}{2}(zd\overline{z} - \overline{z}dz)(X_H)\right) \\
&= z\overline{z} - \left(\overline{z}\frac{\partial}{\partial \overline{z}} - z\frac{\partial}{\partial z}\right) + \left(\frac{i}{2}(zd\overline{z} - \overline{z}dz)i\left(\overline{z}\frac{\partial}{\partial \overline{z}} - z\frac{\partial}{\partial z}\right)\right) \\
&= z\overline{z} - \left(\overline{z}\frac{\partial}{\partial \overline{z}} - z\frac{\partial}{\partial z}\right) - z\overline{z} \\
&= z\frac{\partial}{\partial z} - \overline{z}\frac{\partial}{\partial \overline{z}}.
\end{aligned}$$

Letting the prequantum operator $\widehat{H}$ act on elements of $\mathcal{H}_P$, we see that the partial derivative with respect to $\overline{z}$ vanishes, as the functions in $\mathcal{H}_P$ are holomorphic.
The operator $\widehat{H}$ is a quantum operator, since it preserves the Hilbert space $\mathcal{H}_P$. Indeed, we compute with $\varphi \in \mathcal{H}_P$:

$$\frac{\partial}{\partial \overline{z}}(\widehat{H}\varphi) = \frac{\partial}{\partial \overline{z}}(z\frac{\partial}{\partial z}\varphi) = z\frac{\partial^2 \varphi}{\partial z \partial \overline{z}} = 0,$$

where we have used $\frac{\partial}{\partial \bar{z}} z = 0$ and the symmetry of the partial derivatives.
Hence, the operators $\widehat{z}$, $\widehat{\bar{z}}$ and $\widehat{H}$ are well-defined operator on $\mathcal{H}_P$ which are given by:

$$\widehat{z}\varphi = \frac{z}{2}\varphi,\ \widehat{\bar{z}}\varphi = \frac{3}{2}\frac{\partial \varphi}{\partial z},\ \widehat{H}\varphi = z\frac{\partial \varphi}{\partial z},\ \forall \varphi \in \mathcal{H}_P.$$

We can now calculate the energy spectrum of the quantum harmonic oscillator. Our goal is to find an eigenbasis of $\widehat{H}$ and their corresponding eigenvalues E. We therefore have to solve the time-independent Schrödinger equation $\widehat{H}\varphi = z\frac{\partial}{\partial z}\varphi = E\varphi$. The eigenvector φ is a stationary state of energy E.
We claim that the monomials form the desired eigenbasis. In the 1-dimensional case, monomials are of the form $(z^l)_{l\geq 0}$ and $\widehat{H}$ acts on them by

$$\widehat{H}z^l = z\frac{\partial}{\partial z} z^l = l z^l.$$

These monomials are clearly holomorphic, since $\frac{\partial}{\partial \bar{z}} z^l = 0,\ \forall l \geq 0$. We also calculate the norm of a monomial $z^l,\ l \geq 0$:

$$\begin{aligned}
(z^l \mid z^l) &= \frac{1}{2\pi}\int_{\mathbb{R}^2} |z|^{2l} \exp(-|z|^2) dq \wedge dp \\
&\overset{(1)}{=} \frac{1}{2\pi}\int_{\mathbb{R}>0}\int_0^{2\pi} r^{2l+1}\exp(-r^2) d\theta dr \\
&= \int_{\mathbb{R}>0} r^{2l+1}\exp(-r^2) dr \\
&\overset{(2)}{=} \frac{\Gamma(l+1)}{2} \\
&\overset{(3)}{=} \frac{l!}{2},
\end{aligned} \tag{6.5}$$

where we have used the transformation to polar coordinates in (1), i.e. $z = p + iq = r\exp(i\theta)$, the definition via indefinite integral of the gamma function in (2), i.e. $\Gamma(z) = \int_{\mathbb{R}\geq 0} x^{z-1}\exp(-x)dx$, and the other definition of the gamma function via $\Gamma(l) = (l-1)!$ in (3). Since $\frac{l!}{2} < \infty$, we understand that the monomials are indeed elements of the Segal-Bargmann space $\mathcal{H}_P$. They are also orthogonal with respect to the inner product $(\cdot \mid \cdot)$ on $\mathcal{H}_P$ as we will now show explicitly. Let $m \neq l$ be two integers and $z^m,\ z^l$ be two monomials, we then compute:

$$\begin{aligned}
(z^m \mid z^l) &= \frac{1}{2\pi}\int_{\mathbb{R}^2} z^m \bar{z}^l \exp(-|z|^2) dq \wedge dp \\
&\overset{(1)}{=} \frac{1}{2\pi}\int_{\mathbb{R}\geq 0} r^{m+l+1}\exp(-|r|^2)dr \int_0^{2\pi}\exp(i(m+l)\theta)d\theta \\
&\overset{(2)}{=} 0,
\end{aligned}$$

where we have used again polar coordinates in (1) and the fact that $\exp(i2(m+l)\pi) = 1 = \exp(0)$ in (2). We thus have shown that the monomials define an orthogonal basis of $\mathcal{H}_P$. From equation (6.5), the set $\left(\sqrt{\frac{2}{l!}}z^l\right)_{l\geq 0}$ is an orthonormal basis of $\mathcal{H}_P$ and they form an eigenbasis of the Hamiltonian operator.
The energy $E_l = l$ for monomials of degree l almost correponds (up to the constant $\hbar$) to the known quantum mechanical energies which are $\hbar\left(l+\frac{1}{2}\right)$. The energy shift of $\frac{1}{2}$ is the zero point energy. To correct this issue, one should replace our quantum operator $\widehat{H}$ with the quantum operator $\widehat{H} = z\frac{\partial}{\partial z} + \frac{1}{2}$. This can be achieved with a metaplectic correction which we will not discuss. The interested reader can find more about this topic in [2] Chapter 12.
On the eigenbasis of monomials, we see that the operators $\widehat{z}$ and $\widehat{\overline{z}}$ correspond to the creation and annihilation operators usually denoted by $a^\dagger, a$. Indeed, it is clear that $\widehat{z}$ maps monomials of degree l to monomials of degree $l+1$ and $\widehat{\overline{z}}$ maps monomials of degree l to monomials of degree $l-1$. We also note that the quantum operator of the 1-dimensional harmonic oscillator can be written in terms of these two operators, indeed:

$$\widehat{H} = \frac{4}{3}\widehat{z}\widehat{\overline{z}}.$$

This corresponds exactly to the structure of the quantum harmonic oscillator system, known from quantum mechanics, where $\widehat{H} = a^\dagger a$ up to a constant. We therefore denote $a^\dagger = 2\widehat{z} = z$ and $a = \frac{2}{3}\widehat{\overline{z}} = \frac{\partial}{\partial z}$.
We finally generalise the discussion to the n-dimensional harmonic oscillator. The whole construction is very similar. We therefore won't write the whole derivation again but only the main results. The complexified symplectic form is $\omega = i\sum_{j=1}^n dz_j \wedge d\overline{z}_j$. The quantum Hilbert space, the n-dimensional Segal-Bargmann space, is given by:

$$\mathcal{H}_P := \left\{\varphi \in \mathcal{O}_M \;\middle|\; \frac{1}{2\pi n!}\int_{\mathbb{C}} |\varphi|^2 \exp(-|z|^2)\omega^{\wedge n} < \infty\right\},$$

where $\omega^\wedge n = (i)^n dz_1 \wedge \cdots \wedge dz_n$. The quantum operator of the n-dimensional harmonic oscillator is given by

$$\widehat{H} = \sum_{j=1}^n z_j \frac{\partial}{\partial z_j}.$$

It corresponds to n uncoupled harmonic oscillators. The operators for $\widehat{z}_j$ and $\widehat{\overline{z}}_j$ are given by $\widehat{z}_j = \frac{1}{2}z_j$ and $\widehat{\overline{z}}_j = \frac{3}{2}\frac{\partial}{\partial z_j}$. The eigenbasis is formed by the monomials, that is the set $(z_1^{k_1}z_2^{k_2}\cdots z_n^{k_n} \mid k_1+\cdots+k_n = l)_{l\geq 0}$. We understand that their energies are again given by the degree of the monomial, i.e. $z_1^{k_1}z_2^{k_2}\cdots z_n^{k_n}$ corresponds to the energy $E = k_1+\cdots+k_n$. The operator $\widehat{z}_j$ and $\widehat{\overline{z}}_j$ are now creation and annihilation only for the $z_j^{k_j}$-term, i.e. they correspond to the creation and annihilation operators that we identify as $a_j^\dagger = z_j$ and $a_j = \frac{\partial}{\partial z_j}$. The zero point energy is $\frac{n}{2}$ in the n-dimensional case. With the metaplectic correction, we could find the correct Hamiltonian quantum operator which would take into account this energy shift, i.e. $\widehat{H} = \sum_j z_j \frac{\partial}{\partial z_j} + \frac{n}{2} = \sum_j \frac{4}{3}\widehat{z}_j\widehat{\overline{z}}_j + \frac{n}{2} = \sum_j a_j^\dagger a_j + \frac{n}{2}$.

6.3.3 Segal-Bargmann Transform

We now define the Segal–Bargmann space as a representation of the Weyl algebra and we discuss the Segal–Bargmann transform which will enable us to make sense of the Hamiltonian of the quantum harmonic oscillator in the real polarisation case.
Let us first recall the definition of the Segal–Bargmann space:

Definition 6.3.1. The n-dimensional Segal-Bargmann space $\mathcal{H}L^2(\mathbb{C}^n)$ is the space of holomorphic functions $\varphi \in \mathcal{O}_{\mathbb{C}^n}$ for which

$$\int_{\mathbb{C}^n} |\varphi|^2 \exp(-|z|^2) dz_1 \cdots dz_n < +\infty.$$

As seen in the previous section the monomials

$$(z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n})_{k_1, \cdots, k_n \geq 0}$$

form an orthogonal basis of the Segal–Bargmann space and the creation and annihilation operators $a_j^\dagger = z_j$ and $a_j = \frac{\partial}{\partial z_j}$ are raising and lowering the index of the jth-term.

Proposition 6.3.2. The creation and annihilation operators define a representation $\rho_F : W \to \mathrm{End}(\mathcal{H}L^2(\mathbb{C}^n))$ of the Weyl algebra:

$$\rho_F(X_i) = a_i,\ \rho_F(Y_i) = a_i^\dagger.$$

We call this representation the Fock representation.

Proof. We need to show that the creation and annihilation operators fulfil the canonical commutation relations. We thus compute for $\varphi \in \mathcal{H}L^2(\mathbb{C}^n)$:

$$\begin{aligned}
[a_i, a_j^\dagger]\varphi &= \left[\frac{\partial}{\partial z_i}, z_j\right]\varphi \\
&= \frac{\partial}{\partial z_i} z_j \varphi - z_j \frac{\partial}{\partial z_i}\varphi \\
&\overset{(1)}{=} \delta_{ij}\varphi + z_j \frac{\partial}{\partial z_i}\varphi - z_j \frac{\partial}{\partial z_i}\varphi \\
&= \delta_{ij},
\end{aligned}$$

where we have used the Leibniz rule in (1). □

We therefore know from the Stone–von Neumann theorem (see 4.3.7) that there exists a unitary transformation $\mathcal{B}$ which intertwines the Schrödinger position representation and the Fock representation, i.e. $\rho_F \circ \mathcal{B} = \mathcal{B} \circ \rho_{SP}$. This unitary transformation is called the Segal–Bargmann transform.

Remark 6.3.3. One should actually show an exponentiated form of the commutation relation to use the Stone–von Neumann theorem. The interested reader can find such a discussion in [4] Section 14.4.2.

Theorem 6.3.4. *The Segal–Bargmann transform* $\mathcal{B} : L^2(\mathbb{R}^n) \to \mathcal{H}L^2(\mathbb{C}^n)$ *is defined for all* $\psi \in L^2(\mathbb{R}^n)$ *by:*

$$(\mathcal{B}\psi)(z) := \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}(z \cdot z - 2\sqrt{2}z \cdot x + x \cdot x)\right)\psi(x)dx,$$

where $z \in \mathbb{C}^n$ *and the product* $\cdot$ *is defined as* $a \cdot b = \sum_j a_j b_j$ *for all* $a, b \in \mathbb{C}^n$*.*

Remark 6.3.5. One can actually normalise the Segal–Bargmann transform such that for example the ground state of the harmonic oscillator $\phi_0 \in L^2(\mathbb{R}^n)$ is mapped to $1 \in \mathcal{H}L^2(\mathbb{C}^n)$. This is however not necessary since the quantum system is a projective space and therefore any choice of multiplicative constant in the definition of the Segal–Bargmann transform would yield the same quantum system.

The proof of this theorem can be found in [4] Section 14.4.4. We now apply the Segal–Bargmann transform to the position and momentum operators from $L^2(\mathbb{R}^n)$.

Proposition 6.3.6. The Segal–Bargmann transform intertwines the position and momentum operators $\widehat{q}_j, \widehat{p}_j$ in the following way:

$$\begin{aligned} \mathcal{B}\left(\frac{\widehat{q}_j + i\widehat{p}_j}{\sqrt{2}}\right)\mathcal{B}^{-1} &= a_j \\ \mathcal{B}\left(\frac{\widehat{q}_j - i\widehat{p}_j}{\sqrt{2}}\right)\mathcal{B}^{-1} &= a_j^\dagger. \end{aligned} \tag{6.6}$$

Proof (Sketch). We restrict ourselves to the 1-dimensional case as the calculation is analogous for the n-dimensional case. We show (6.6) for smooth and rapidly decaying functions $\psi \in L^2(\mathbb{R})$. Up to multiplying $\mathcal{B}$ on the right, proving (6.6) is equivalent to showing:

$$\begin{aligned} \mathcal{B}\frac{\widehat{q} + i\widehat{p}}{\sqrt{2}} &= \frac{\partial}{\partial z}\mathcal{B} \\ \mathcal{B}\frac{\widehat{q} - i\widehat{p}}{\sqrt{2}} &= z\mathcal{B}. \end{aligned} \tag{6.7}$$

We now compute the right hand side of the first equation of 6.7:

$$\begin{aligned} \frac{\partial}{\partial z}(\mathcal{B}\psi)(z) &\overset{(1)}{=} \int_{\mathbb{R}^n} \frac{\partial}{\partial z}\exp\left(-\frac{1}{2}(z \cdot z - 2\sqrt{2}z \cdot x + x \cdot x)\right)\psi(x)dx \\ &= \int_{\mathbb{R}^n} (-z + \sqrt{2}x)\exp\left(-\frac{1}{2}(z \cdot z - 2\sqrt{2}z \cdot x + x \cdot x)\right)\psi dx \\ &= \sqrt{2}\mathcal{B}(x\psi)(z) - z(\mathcal{B}\psi)(z), \end{aligned}$$

where we can differentiate under the integral in (1) because of the assumptions on ψ. This means that

$$\frac{\partial}{\partial z}\mathcal{B} = \sqrt{2}\mathcal{B}x - z\mathcal{B}. \tag{6.8}$$

We then compute the image of the derivative under the Segal–Bargmann transform:

$$\begin{aligned}
\mathcal{B}\left(\frac{\partial \psi}{\partial x}\right)(z) &= \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}(z\cdot z - 2\sqrt{2}z\cdot x + x\cdot x)\right)\frac{\partial \psi}{\partial x}dx \\
&\overset{(1)}{=} -\int_{\mathbb{R}^n} (\sqrt{2}z - x)\exp\left(-\frac{1}{2}(z\cdot z - 2\sqrt{2}z\cdot x + x\cdot x)\right)\psi dx \\
&= \mathcal{B}(x\psi)(z) - \sqrt{2}z(\mathcal{B}\psi)(z),
\end{aligned}$$

where in (1) integration by part is used and the boundary term vanishes because of the assumptions on ψ. Therefore

$$\mathcal{B}\frac{\partial}{\partial x} = \mathcal{B}x - \sqrt{2}z\mathcal{B}. \tag{6.9}$$

We can now combine both results. First, we replace $z\mathcal{B}$ in equation (6.8) with what was found in equation (6.9) and we find:

$$\frac{\partial}{\partial z}\mathcal{B} = \mathcal{B}\frac{1}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right) = \mathcal{B}\frac{\widehat{q} - i\widehat{p}}{\sqrt{2}},$$

which is the first equation in (6.6). We can then replace $\frac{\partial}{\partial z}\mathcal{B}$ and isolate $z\mathcal{B}$ in equation (6.8). We find:

$$z\mathcal{B} = \mathcal{B}\frac{1}{\sqrt{2}}\left(x - \frac{\partial}{\partial x}\right) = \mathcal{B}\frac{\widehat{q} + i\widehat{p}}{\sqrt{2}},$$

which is the second equation in (6.6). □

We now transform the Hamiltonian of the quantum harmonic oscillator $\widehat{H}_{SB}$ found in the Segal-Bargmann space into the Schrödinger position representation, which we will denote $\widehat{H}_{SP}$. We recall that the Hamiltonian in the Segal–Bargmann space was given by $\widehat{H}_{SB} = \sum_j a_j^\dagger a_j$ (neglecting the constant $\frac{4}{3}$ because it is dynamically irrelevant). We now know how the Segal–Bargmann transform maps this Hamiltonian operator. Indeed, we can compute:

$$\begin{aligned}
\widehat{H}_{SP} &= \mathcal{B}^{-1}\widehat{H}_{SB}\mathcal{B} \\
&= \mathcal{B}^{-1}\sum_j a_j^\dagger a_j \mathcal{B} \\
&= \sum_j \mathcal{B}^{-1} a_j^\dagger \mathcal{B}\mathcal{B}^{-1} a_j \mathcal{B} \\
&\overset{(1)}{=} \sum_j \left(\frac{\widehat{q}_j - i\widehat{p}_j}{\sqrt{2}}\right)\left(\frac{\widehat{q}_j + i\widehat{p}_j}{\sqrt{2}}\right) \\
&= \sum_j \frac{\widehat{q}_j^2 + \widehat{p}_j^2}{2} + \frac{i}{2}(\widehat{q}_j\widehat{p}_j - \widehat{p}_j\widehat{q}_j) \\
&\overset{(2)}{=} -\frac{n}{2} + \sum_j \frac{\widehat{q}_j^2 + \widehat{p}_j^2}{2},
\end{aligned}$$

where we have used equations (6.6) in (1) and the commutation relation $\widehat{q}_j\widehat{p}_j - \widehat{p}_j\widehat{q}_j = [\widehat{q}_j, \widehat{p}_j] = i$ in (2). With the metaplectic correction, we would get:

$$\widehat{H}_{SP} = \mathcal{B}^{-1}\widehat{H}_{SB}\mathcal{B} = \mathcal{B}^{-1}\sum_j a_j^\dagger a_j \mathcal{B} + \mathcal{B}^{-1}\left(\frac{n}{2}\right)\mathcal{B} = \sum_j \frac{\widehat{q}_j^2 + \widehat{p}_j^2}{2}.$$

This corresponds to the Hamiltonian of the quantum harmonic oscillator in the Schrödinger position representation (up to some coefficient that we have neglected, see discussion at the beginning of 6.3.1) known from quantum mechanics, see Example 4.4.3. The energy spectrum is then exactly given by $E_l = l + \frac{n}{2},\ l \in \mathbb{N}$, as desired.

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